KOSZUL DUALITY OF AFFINE KAC-MOODY ALGEBRAS AND CYCLOTOMIC RATIONAL DOUBLE AFFINE HECKE ALGEBRAS

P. SHAN, M. VARAGNOLO, E. VASSEROT

Abstract. We give a proof of the parabolic/singular Koszul duality for the category O of affine Kac-Moody algebras. The main new tool is a relation between moment graphs and finite codimensional affine Schubert varieties. We apply this duality to q-Schur algebras and to cyclotomic rational double affine Hecke algebras.

Contents

1. Introduction

The purpose of this paper is to give a proof of the parabolic/singular Koszul duality for the category O of affine Kac-Moody algebras. The main motivation for this is the conjecture in [41] relating the parabolic affine category O and the category O of cyclotomic rational double affine Hecke algebras (CRDAHA for short).

They are several possible approaches to Koszul duality for affine Kac-Moody algebras. In [6], a geometric analogue of the composition of the Koszul and the Ringel duality is given, which involves Whittaker sheaves on the affine flag variety. Our principal motivation comes from CRDAHA's. For this, we need to prove Koszul duality for the category O itself reather than for its geometric analogues.

One difficulty of the Kac-Moody case comes from the fact that, at a positive level, the category O has no tilting module, while at a negative level it has no projective module. One way to overcome this is to use a different category of modules than the usual category O, as the Whittaker category in loc. cit. or a category of linear complexes as in [31]. Since we want a version of Koszul duality which we can apply to q-Schur algebras and CRDAHA's, we use a different point of view. Under truncation the affine, parabolic, singular category O at a non-critical level yields a finite highest-weight category which contains both tilting and projective objects. We prove that these highest weight categories are Koszul and are Koszul dual to each other.

Another difficulty comes from the absence of a localization theorem (from the category O to perverse sheaves) at the positive level. To overcome this we use standard Koszul duality. See Section 2.6 below for details.

Our general argument is similar to the one in [5] : we use the affine analogue of the Soergel functor introduced in [14]. It uses the deformed category O, which is a highest weight category over a localization of a polynomial ring. It also uses some category of sheaves over a moment graph. Note that the affine category O is related to two different types of geometry. In negative level it is related to the affine flag ind-scheme and to finite dimensional affine Schubert varieties. In positive level it is related to Kashiwara's affine flag manifold and to finite codimensional affine Schubert varieties. An important new tool in our work is a relation between sheaves over some moment graph and equivariant perverse sheaves on finite codimensional affine Schubert varieties. This relation is of independent interest.

Next, we apply this Koszul duality to q-Schur algebras. The Kazhdan-Lusztig equivalence [27] implies that the module category of the q-Schur algebra is equivalent to a highest weight subcategory of the affine category O of GL_n at a negative level. Thus, our result implies that the q-Schur-algebra is Koszul (and also standard Koszul), see Remark B.6[1]. To our knowledge, this was not proved so far. There are different possible approachs for proving that the q-Schur algebra is Koszul. Some are completely algebraic, see e.g., [33]. Some use analogues of the Bezrukavnikov-Mirkovic modular localization theorem, see e.g., [34]. Our approach has the advantage that it yields an explicit description of the Koszul dual of the q-Schur algebra.

Finally, we also apply this duality to CRDAHA's. More precisely, in [41] some higher analogue of the q-Schur algebra has been introduced. It is a highest-weight subcategory of the parabolic category O of an affine Kac-Moody algebra at a negative level. It is conjectured in loc. cit. that these higher q-Schur algebras are equivalent to the category O of the CRDAHA. Our result implies that these higher q-Schur algebras are Koszul and are Koszul dual to each other. This result should

[1] After our paper was written, we received a copy of [9] where a similar result is obtained by different methods

be regarded as an analogue of a conjectural Koszulity of the CRDAHA, see e.g., [9].

2. Preliminaries

2.1. **Categories.** For an object M of a category $\mathbf{C}$ let $\mathbf{1}_M$ be the identity endomorphism of M. Let $\mathbf{C}^{\mathrm{op}}$ be the category opposite to $\mathbf{C}$.

If $\mathbf{C}$ is an exact category then $\mathbf{C}^{\mathrm{op}}$ is equip with the exact structure such that $0 \to M' \to M \to M'' \to 0$ is exact in $\mathbf{C}^{\mathrm{op}}$ if and only if $0 \to M'' \to M \to M' \to 0$ is exact in $\mathbf{C}$. An *exact* functor of exact categories is a functor which takes short exact sequences to short exact sequences. A contravariant functor $F : \mathbf{C}' \to \mathbf{C}$ is exact if the functor $F : \mathbf{C}' \to \mathbf{C}^{\mathrm{op}}$ is exact.

Let $\mathbf{C}$ be an abelian category. Let $\mathrm{Irr}(\mathbf{C})$ be the set of isomorphism classes of simple objects and let $\mathrm{Proj}(\mathbf{C})$ be the set of isomorphism classes of indecomposable projective objects. For an object M of $\mathbf{C}$ we abbreviate $\mathrm{Ext}_{\mathbf{C}}(M) = \mathrm{Ext}_{\mathbf{C}}(M, M)$.

Let A be a commutative, noetherian, integral domain. An *A-category* is an additive category enriched over the tensor category of A-modules. A *graded A-category* is an additive category enriched over the monoidal category of graded A-modules. Unless mentioned otherwise, a functor of A-categories is always assumed to be A-linear. A *Hom-finite A-category* is an A-category whose Hom spaces are finitely generated over A.

An additive category is *Krull-Schmidt* if any object has a decomposition such that each summand is indecomposable with local endomorphism ring. A full additive subcategory of a Krull-Schmidt category is again Krull-Schmidt if and only if every idempotent splits.

If $A = k$ is a field a Hom-finite k-category is Krull-Schmidt if and only if every idempotent splits. In particular a Hom-finite exact k-category is Krull-Schmidt. A *finite abelian k-category* is a Hom-finite abelian k-category whose objects have a finite length. It is equivalent to the category of finite dimensional modules over a finite dimensional k-algebra if and only if it admits a projective generator.

2.2. **Graded rings.** For a ring R let $\mathbf{Mod}(R)$ be the category of all R-modules and let $\mathbf{mod}(R)$ be the category of the finitely generated ones. We abbreviate

$$\mathrm{Irr}(R) = \mathrm{Irr}(\mathbf{mod}(R)), \quad \mathrm{Proj}(R) = \mathrm{Proj}(\mathbf{mod}(R)).$$

By a graded ring $\bar{R}$ we'll always mean a $\mathbb{Z}$-graded ring. Let $\mathbf{gmod}(\bar{R})$ be the category of the finitely generated graded $\bar{R}$-modules. We abbreviate

$$\mathrm{Irr}(\bar{R}) = \mathrm{Irr}(\mathbf{gmod}(\bar{R})), \quad \mathrm{Proj}(\bar{R}) = \mathrm{Proj}(\mathbf{gmod}(\bar{R})).$$

Given a graded $\bar{R}$-module M and an integer j, let $M\{j\}$ be the graded $\bar{R}$-module obtained from M by shifting the grading by j, i.e., such that $M\{j\}^i = M^{i+j}$. The ring $\bar{R}$ is *positively graded* if $\bar{R}^{<0} = 0$ and $\bar{R}^{>0}$ is the radical of $\bar{R}$, e.g., a finite dimensional graded algebra $\bar{R}$ over a field k is positively graded if $\bar{R}^{<0} = 0$ and $\bar{R}^0$ is semisimple as a $\bar{R}$-module. Here $\bar{R}^0$ is identified with $\bar{R}/\bar{R}^{>0}$.

Assume that k is a field and that $\bar{R}$ is a positively graded finite dimensional k-algebra. We say that $\bar{R}$ is *basic* if $\bar{R}^0$ isomorphic to a finite product of copies of k as a k-algebra. Let $\{1_x\}$ be a complete system of primitive orthogonal idempotents of $\bar{R}^0$. The *Hilbert polynomial* of $\bar{R}$ is the matrix $P(\bar{R}, t)$ with entries

$$P(\bar{R}, t)_{x,x'} = \sum_i t^i \dim\big(1_x \bar{R}^i 1_{x'}\big) \in \mathbb{N}[[t]].$$

2.3. **Koszul duality.** Let k be a field and $\bar{R}$ be a positively graded finite dimensional k-algebra. The *Koszul dual* of $\bar{R}$ is the graded k-algebra

$$E(\bar{R}) = \mathrm{Ext}_R(\bar{R}^0),$$

where R is the (non graded) k-algebra underlying $\bar{R}$. Forgetting the grading, we get a k-algebra $E(R)$. Assume that $\bar{R}$ is *Koszul*. Then $E(\bar{R})$ is also Koszul, we have $E^2(\bar{R}) = \bar{R}$, and there is a natural contravariant equivalence of triangulated categories

$$E : \mathbf{D}^b(\mathbf{gmod}(\bar{R})) \to \mathbf{D}^b(\mathbf{gmod}(E(\bar{R})))$$

which takes simple graded modules to projective ones. Let $\{1_x\,;\, x \in \mathrm{Irr}(\bar{R}^0)\}$ be a complete system of primitive orthogonal idempotents of $\bar{R}^0$. Note that

$$\mathrm{Irr}(R) = \mathrm{Irr}(\bar{R}^0) = \{\bar{R}^0 1_x\}.$$

Via the canonical bijection $\mathrm{Proj}(R) \simeq \mathrm{Irr}(R)$, the elements 1_x, $x \in \mathrm{Irr}(\bar{R}^0)$, can be viewed as a complete system of primitive orthogonal idempotents of R. The functor E yields a bijection

$$\phi : \mathrm{Irr}(R) \to \mathrm{Proj}(E(R)).$$

We set $E(1_x) = 1_{\phi(\bar{R}^0 1_x)}$. The elements $E(1_x)$, $x \in \mathrm{Irr}(R)$, form a complete system of primitive orthogonal idempotents of $E(R)$. We may abbreviate $1_x = E(1_x)$. Via the canonical bijection $\mathrm{Proj}(E(R)) \simeq \mathrm{Irr}(E(R))$ the map ϕ can be viewed as a bijection $\phi : \mathrm{Irr}(R) \to \mathrm{Irr}(E(R))$. We'll call it the *natural bijection*.

If $\bar{R}$ is Koszul we say that R *has a Koszul grading*. If R has a Koszul grading then this grading is unique up to isomorphism of graded k-algebras, see [5, cor. 2.5.2].

Let $\mathbf{C}$ be a finite abelian k-category with a projective generator P. We say that $\mathbf{C}$ *has a Koszul grading* if $R = \mathrm{End}_{\mathbf{C}}(P)^{\mathrm{op}}$ has a Koszul grading. The following lemmas are well-known, see e.g., [5].

Lemma 2.1. *Let P, R, $\mathbf{C}$ be as above. If $\bar{R}$ is a positively graded k-algebra then $E(\bar{R}) = \mathrm{Ext}_{\mathbf{C}}(L)$, where L is the top of P.*

Lemma 2.2. *Let $\mathbf{C}$ be a finite abelian k-category with a Koszul grading. If $\mathbf{C}'$ is a thick subcategory and the inclusion $\mathbf{C}' \subset \mathbf{C}$ induces injections on extensions, then $\mathbf{C}'$ has also a Koszul grading.*

2.4. **Highest weight categories.** Let A be a commutative, noetherian, integral domain which is a local ring with residue field k. Let K be its fraction field. Note that any finitely generated projective A-module is free. Let $\mathbf{C}$ be an A-category equivalent to the category of finitely generated modules over a finite projective A-algebra R. Assume that $\mathbf{C}$ is of highest-weight over A, see [36, def. 4.11]. The sets $\Delta(\mathbf{C})$, $\nabla(\mathbf{C})$ of isomorphisms classes of standard and costandard objects are uniquely defined by [36, rk. 4.17, prop. 4.19]. Let $\mathbf{C}^\Delta$, $\mathbf{C}^\nabla$ be the full subcategories of $\mathbf{C}$ consisting of the Δ-*filtered* and ∇-*filtered* objects, i.e., the objects having a finite filtration whose successive quotients are standard, costandard respectively. These categories are exact. Recall that the opposite of $\mathbf{C}$ is a highest weight category such that $\Delta(\mathbf{C}^{\mathrm{op}}) = \nabla(\mathbf{C})$ with the opposite order. Let $\mathrm{Tilt}(\mathbf{C})$ be the set of isomorphism classes of indecomposable tilting objects.

For an A-module M we write $kM = M \otimes_A k$. Write also $k\mathbf{C} = \mathbf{mod}(kR)$. We call the functor $M \mapsto kM$ the *reduction to* k. The k-category $k\mathbf{C}$ is a highest weight category [36, thm. 4.15] and the reduction to k yields bijections

$$\Delta(\mathbf{C}) \to \Delta(k\mathbf{C}), \quad \nabla(\mathbf{C}) \to \nabla(k\mathbf{C}), \quad \mathrm{Irr}(\mathbf{C}) \to \mathrm{Irr}(k\mathbf{C}).$$

We have also canonical bijections

$$\Delta(\mathbf{C}) \simeq \nabla(\mathbf{C}) \simeq \mathrm{Irr}(\mathbf{C}).$$

We call *tilting generator* a tilting object which is also a *tilting complex*, see e.g., [36, sec. 4.1.5]. We call *full tilting module* a minimal tilting generator. We call *full projective module* a minimal projective generator. We have the following analogue of [13, prop. 2.6].

Proposition 2.3. *Let $K\mathbf{C}$ be split semisimple.*

(a) The reduction to k gives bijections

$$\mathrm{Proj}(\mathbf{C}) \to \mathrm{Proj}(k\mathbf{C}), \quad \mathrm{Tilt}(\mathbf{C}) \to \mathrm{Tilt}(k\mathbf{C}). \tag{2.1}$$

(b) There are natural bijections

$$\mathrm{Tilt}(\mathbf{C}) = \mathrm{Proj}(\mathbf{C}) = \Delta(\mathbf{C}) = \nabla(\mathbf{C}) = \mathrm{Irr}(\mathbf{C}). \tag{2.2}$$

Proof. An object $M \in \mathbf{C}$ is projective if and only if it is projective over A and kM is projective over kR. By [11, sec. 6, ex. 16], an object $M \in \mathbf{C}$ is indecomposable if and only if kM is indecomposable in $k\mathbf{C}$. Thus the reduction to k gives a map from $\mathrm{Proj}(\mathbf{C})$ to $\mathrm{Proj}(k\mathbf{C})$. It takes obviously a projective generator of $\mathbf{C}$ to a projective generator of $k\mathbf{C}$. Thus it gives a bijection $\mathrm{Proj}(\mathbf{C}) \to \mathrm{Proj}(k\mathbf{C})$. Finally, by [36, prop. 4.26], the reduction to k takes a tilting generator of $\mathbf{C}$ to a tilting generator of $k\mathbf{C}$. So we get also a bijection $\mathrm{Tilt}(\mathbf{C}) \to \mathrm{Tilt}(k\mathbf{C})$. This proves (a). Part (b) follows from (a), because (b) is obviously true for a highest weight category over a field. ☐

Remark 2.4. If (2.2) holds then any of the sets $\mathrm{Tilt}(\mathbf{C})$, $\mathrm{Proj}(\mathbf{C})$, $\Delta(\mathbf{C})$, $\nabla(\mathbf{C})$, $\mathrm{Irr}(\mathbf{C})$ can be regarded as a poset for the highest weight order.

Remark 2.5. An *ideal* of a poset $(S, \leqslant)$ is a subset I such that $I = \bigcup_{i \in I}\{\leqslant i\}$. A *coideal* is the complement of an ideal. For a subset $I \subset \mathrm{Irr}(\mathbf{C})$, let $\mathbf{C}[I]$ be the thick subcategory generated by I and let $\mathbf{C}(I)$ be the Serre quotient $\mathbf{C}/\mathbf{C}[\mathrm{Irr}(\mathbf{C}) \setminus I]$. Assume that $\mathbf{C}$ is a highest weight category over a field k, and that I, J are an ideal and a coideal of $(\mathrm{Irr}(\mathbf{C}), \leqslant)$. Then $\mathbf{C}[I]$, $\mathbf{C}(J)$ are highest weight categories and the inclusion $\mathbf{C}[I] \subset \mathbf{C}$ induces injections on extensions by [10, thm. 3.9], [12, prop. A.3.3].

2.5. **Ringel duality.** Let A be a commutative, noetherian, integral domain which is a local ring with residue field k. Let $\mathbf{C}$ be a highest-weight category over A which is equivalent to the category of finitely generated modules over a finite projective A-algebra R. Let T be a tilting generator. The *Ringel dual* of R is the A-algebra $D(R) = \mathrm{End}_{\mathbf{C}}(T)^{\mathrm{op}}$, the Ringel dual of $\mathbf{C}$ is the category $D(\mathbf{C}) = \mathbf{mod}(D(R))$. The category $D(\mathbf{C})$ is a highest-weight category over A with

$$\Delta(D(\mathbf{C})) = \{\mathrm{Hom}_{\mathbf{C}}(T, \nabla)\,;\, \nabla \in \nabla(\mathbf{C})\} \simeq \Delta(\mathbf{C}).$$

The order on $\Delta(D(\mathbf{C}))$ is the opposite of the order on $\Delta(\mathbf{C})$. We have an exact contravariant equivalence called the *tilting equivalence*

$$D:\ \mathbf{C}^{\Delta} \to D(\mathbf{C})^{\Delta},\ M \mapsto \mathrm{Hom}_{\mathbf{C}}(M, T).$$

It takes tilting objects to projective ones, and projective objects to tilting ones. The algebra $D^2(R)$ is Morita equivalent to R. See [36, prop. 4.26], [12, sec. A.4].

Assume that (2.2) holds for $\mathbf{C}$, $D(\mathbf{C})$. Then the tilting equivalence yields bijections $\psi : \mathrm{Proj}(\mathbf{C}) \to \mathrm{Tilt}(D(\mathbf{C})) \simeq \mathrm{Proj}(D(\mathbf{C}))$. So, we get a bijection

$$\{1_x\,;\, x \in \mathrm{Proj}(R)\} \to \{1_x\,;\, x \in \mathrm{Proj}(D(R))\}, \quad 1_x \mapsto D(1_x) = 1_{\psi(x)}.$$

Remark 2.6. The canonical map $k\,\mathrm{Hom}_{\mathbf{C}}(M, N) \to \mathrm{Hom}_{k\mathbf{C}}(kM, kN)$ is invertible for any R-modules M, N which are free of finite type over A [36, prop. 4.30]. Thus, the tilting equivalence commutes with the reduction to k.

Remark 2.7. Let $\mathbf{C}$ be a highest-weight category over a field k. Let $I \subset \mathrm{Irr}(\mathbf{C})$ be an ideal. We may regard to ψ as a map $\mathrm{Irr}(\mathbf{C}) \to \mathrm{Irr}(D(\mathbf{C}))$ in the obvious way. Then $\psi(I)$ is a coideal of $\mathrm{Irr}(D(\mathbf{C}))$ and the tilting equivalence $D:\ \mathbf{C}^\Delta \to D(\mathbf{C})^\Delta$ factors to an equivalence $\mathbf{C}[I]^\Delta \to D(\mathbf{C})(\psi(I))^\Delta$. It induces an equivalence of highest weight categories $D(\mathbf{C}[I]) \simeq D(\mathbf{C})(\psi(I))$, see e.g., [12, prop. A.4.9].

2.6. **Standard Koszul duality.** Let A be a commutative, noetherian, integral domain which is a local ring with residue field k. Assume that A is positively graded. Let $\mathbf{C}$ be a highest weight category over A which is equivalent to the category of finitely generated modules over a finite projective A-algebra R. Let $\bar{R}$ be a positively graded A-algebra which is isomorphic to R as an A-algebra. We call *graded lift* of an object M of $\mathbf{C}$ a $\bar{R}$-module which is isomorphic to M as a R-module. Recall the following lemma.

Lemma 2.8. *Let M be an indecomposable R-module which is finite projective over A and such that kM is indecomposable over kR. If M has a graded lift then this lift is unique up to a graded $\bar{R}$-module isomorphism and up to a shift of the grading.*

Proof. If $A = k$ a proof is given in [5, lem. 2.5.3]. For the general case it is enough to check that $\mathrm{End}_R(M)$ is a local ring. This is obvious, because, since M is finite projective over A, an element $x \in \mathrm{End}_R(M)$ is invertible if and only if is reduction to k is invertible in $\mathrm{End}_{kR}(kM)$ by the Nakayama lemma. $\square$

Assume that (2.1) holds. Objects of $\mathrm{Irr}(\mathbf{C})$ have graded lifts which are pure of degree 0. Objects of $\mathrm{Proj}(\mathbf{C})$ have graded lifts such that the projection to their top is homogeneous of degree 0. Objects of $\Delta(\mathbf{C})$ have graded lifts such that the projection from their projective cover is homogeneous of degree 0. The proof is the same as in the case where A is a field, see e.g., [29, cor. 4]. Finally, objects of $\mathrm{Tilt}(\mathbf{C})$ have graded lifts such that the inclusion of the highest standard object is homogeneous of degree 0. The proof is the same as in the case where A is a field, see e.g., [29, cor. 5], using the construction of tilting modules in [36, prop. 4.26] instead of Ringel's construction in [35]. The gradings above are called the *natural gradings*. Note that the natural grading commutes with the reduction to k.

The natural grading on the full tilting module gives a grading on $D(R)$ which is called again the *natural grading*. Let $D(\bar{R})$ denote the A-algebra $D(R)$ with its natural grading. Assume that $D(\bar{R})$ is a positively graded A-algebra. The contravariant functor $M \mapsto \mathrm{Hom}_R(M, \bar{T})$ takes the natural graded indecomposable tilting objects to the natural graded indecomposable projective ones. It takes also the natural graded indecomposable projective objects to natural graded indecomposable tilting ones.

Now, let $A = k$. Let $\leqslant$ be the highest weight order on $\{1_x\,;\, x \in \mathrm{Irr}(R)\}$. Following [2, thm. 1.4, 3.4], we say that $\bar{R}$ is *standard Koszul* provided that $E(\bar{R})$ is quasi-hereditary relatively to the poset $(\{E(1_x)\,;\, x \in \mathrm{Irr}(R)\}, \geqslant)$. Following [29, thm. 7], we say that $\bar{R}$ is *balanced* if it is standard Koszul and if the graded k-algebra $D(\bar{R})$ is positive. If $\bar{R}$ is balanced then the following holds [30, thm. 1]

- $\bar{R}$ is Koszul and standard Koszul,
- $\bar{R}$, $D(\bar{R})$, $E(\bar{R})$ and $DE(\bar{R})$ are positively graded, quasi-hereditary, Koszul and balanced,
- $DE(\bar{R}) = ED(\bar{R})$ as graded quasi-hereditary k-algebras,
- the natural bijection $\phi : \mathrm{Irr}(R) \to \mathrm{Irr}(E(R))$ takes the highest weight order on $\mathrm{Irr}(R)$ to the opposite of the highest weight order on $\mathrm{Irr}(E(R))$.

Remark 2.9. Assume that $\mathbf{C}$ is a highest weight category over k which is balanced. Let $I \subset \mathrm{Irr}(\mathbf{C})$ be an ideal. The category $\mathbf{C}[I]$ has a Koszul grading by Lemma 2.2

and Remark 2.5. Let $\phi : \mathrm{Irr}(\mathbf{C}) \to \mathrm{Irr}(E(\mathbf{C}))$ be the natural bijection. Note that $\phi(I)$ is a coideal of $\mathrm{Irr}(E(\mathbf{C}))$. By Lemma 2.1 and Remark 2.5 we have $E(\mathbf{C}[I]) = E(\mathbf{C})(\phi(I))$.

2.7. **Lie algebras.** Let $\mathfrak{g}$ be a simple Lie $\mathbb{C}$-algebra and let G be a connected simple group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. Let $T \subset G$ and $\mathfrak{t} \subset \mathfrak{g}$ be maximal tori, with $\mathrm{Lie}(T)=\mathfrak{t}$. Let $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra containing $\mathfrak{t}$. The elements of $\mathfrak{t}$, $\mathfrak{t}^*$ are called *coweights* and *weights* respectively. Given a root $\alpha \in \mathfrak{t}^*$ let $\check\alpha \in \mathfrak{t}$ denote the corresponding coroot. Let ρ be half the sum of the positive roots. Let $\Pi \subset \mathfrak{t}^*$ be the set of roots, $\Pi^+ \subset \Pi$ the set of positive roots and $\mathbb{Z}\Pi$ be the root lattice. Let $\Phi = \{\alpha_i\,;\, i \in I\}$ be the set of simple roots in Π^+. Let W be the Weyl group. Let m be the dual Coxeter number of $\mathfrak{g}$.

2.8. **Affine Lie algebras.** Let $\mathbf{g}$ be the affine Lie algebra associated with $\mathfrak{g}$. Recall that $\mathbf{g} = \mathbb{C}\partial \oplus \widehat{L\mathfrak{g}}$, where $\widehat{L\mathfrak{g}}$ is a central extension of $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ and $\partial = t\partial_t$ is a derivation of $\widehat{L\mathfrak{g}}$ acting trivially on the canonical central element $\mathbf{1}$ of $\widehat{L\mathfrak{g}}$. Consider the Lie subalgebras

$$\mathbf{b} = \mathbb{C}\partial \oplus L\mathfrak{b} \oplus \mathbb{C}\mathbf{1}, \quad \mathbf{t} = \mathbb{C}\partial \oplus \mathfrak{t} \oplus \mathbb{C}\mathbf{1}.$$

The elements of $\mathbf{t}$, $\mathbf{t}^*$ are called *affine coweights* and *affine weights* respectively. Let $(\bullet : \bullet) : \mathbf{t}^* \times \mathbf{t} \to \mathbb{C}$ be the canonical pairing, let $\widehat\Pi$ be the set of roots of $\mathbf{g}$ and let $\widehat\Pi^+$ be the set of roots of $\mathbf{b}$. We set $\widehat\Pi^- = -\widehat\Pi^+$. We'll call an element of $\widehat\Pi$ an *affine root*. The set of simple roots in $\widehat\Pi^+$ is $\widehat\Phi = \{\alpha_i; i \in \{0\} \cup I\}$. Let $\check\alpha \in \mathbf{t}$ be the affine coroot associated with the real affine root α. Let δ, Λ_0, $\hat\rho$ be the affine weights given by

$$(\delta : \partial) = (\Lambda_0 : \mathbf{1}) = 1, \quad (\Lambda_0 : \mathbb{C}\partial \oplus \mathfrak{t}) = (\delta : \mathfrak{t} \oplus \mathbb{C}\mathbf{1}) = 0, \quad \hat\rho = \rho + m\Lambda_0.$$

We'll use the identification $\mathbf{t}^* = \mathbb{C} \times \mathfrak{t}^* \times \mathbb{C}$ such that $\alpha_i \mapsto (0, \alpha_i, 0)$ if $i \neq 0$, $\Lambda_0 \mapsto (0, 0, 1)$ and $\delta \mapsto (1, 0, 0)$. An element of $\mathbf{t}^*/\mathbb{C}\delta$ is called a *classical affine weight*. Let $cl : \mathbf{t}^* \to \mathbf{t}^*/\mathbb{C}\delta$ denote the obvious projection. Let $\langle \bullet : \bullet \rangle$ be the non-degenerate symmetric bilinear form on $\mathbf{t}^*$ such that

$$(\lambda : \check\alpha_i) = 2\langle \lambda : \alpha_i \rangle / \langle \alpha_i : \alpha_i \rangle, \quad (\lambda : \mathbf{1}) = \langle \lambda : \delta \rangle.$$

Using $\langle \bullet : \bullet \rangle$ we identify $\check\alpha$ with an element of $\mathbf{t}^*$ for any real affine root α. Let $\widehat W = W \ltimes \mathbb{Z}\Pi$ be the affine Weyl group and let $\mathcal{S} = \{s_i = s_{\alpha_i}\,;\, \alpha_i \in \widehat\Phi\}$ be the set of simple affine reflections. The group $\widehat W$ acts on $\mathbf{t}^*$. For $w \in W$, $\tau \in \mathbb{Z}\Pi$ we have

$$w(\Lambda_0) = \Lambda_0, \quad w(\delta) = \delta, \quad \tau(\delta) = \delta,$$
$$\tau(\lambda) = \lambda - \langle \tau : \lambda \rangle \delta, \quad \tau(\Lambda_0) = \tau + \Lambda_0 - \langle \tau : \tau \rangle \delta/2.$$

The $\bullet$-*action* on $\mathbf{t}^*$ is given by $w \bullet \lambda = w(\lambda + \hat\rho) - \hat\rho$. This action factors to a $\widehat W$-action on $\mathbf{t}^*/\mathbb{C}\delta$. Two (classical) affine weights λ, μ are *linked* if they belong to the same orbit of the $\bullet$-action, and we write $\lambda \sim \mu$. Let W_λ be the stabilizer of an affine weight λ. We say that λ is *regular* if $W_\lambda = \{1\}$. For $e \in \mathbb{C}$ we set

$$\mathbf{t}^*_e = \{\lambda \in \mathbf{t}^*; (\lambda : \mathbf{1}) = -e - m\}.$$

From now on we assume that e is an integer $\neq 0$. Set

$$\mathcal{C}^\pm = \{\lambda \in \mathbf{t}^*\,;\, \langle \lambda + \hat\rho : \alpha \rangle \geqslant 0,\, \alpha \in \widehat\Pi^\pm\}.$$

An element of $\mathcal{C}^-$, resp. of $\mathcal{C}^+$, is called an *antidominant affine weight*, resp. a *dominant affine weight*. We write again $\mathcal{C}^\pm$ for $cl(\mathcal{C}^\pm)$. For any integral affine weight λ of level $-e-m$ we have

$$\begin{cases} \sharp(\widehat W \bullet \lambda \cap \mathcal{C}^-) = 1, & \sharp(\widehat W \bullet \lambda \cap \mathcal{C}^+) = 0, \qquad \text{if } e > 0, \\ \sharp(\widehat W \bullet \lambda \cap \mathcal{C}^+) = 1, & \sharp(\widehat W \bullet \lambda \cap \mathcal{C}^-) = 0 \qquad \text{if } e < 0, \end{cases}$$

see e.g., [23, lem. 2.10]. In the first case we say that λ is *positive*, in the second one that it is *negative*. For $\lambda \in \mathcal{C}^{\pm}$ the subgroup W_λ of $\widehat{W}$ is finite and parabolic. It is isomorphic to the Weyl group of the root system $\{\alpha \in \widehat{\Pi}\,;\, \langle \lambda + \hat{\rho} : \alpha \rangle = 0\}$.

2.9. **Parabolic category O.** Let $\mathcal{P}$ be the set of proper subsets of $\widehat{\Phi}$. An element of $\mathcal{P}$ is called a *parabolic type*. Fix a parabolic type ν. If ν is the empty set we say that ν *is regular*, and we write $\nu = \phi$. Let $\mathbf{p}_\nu \subset \mathbf{g}$ be the corresponding parabolic subalgebra containing $\mathbf{b}$. Let Π_ν be the root system of a levi of $\mathbf{p}_\nu$, and let $\Pi_\nu^+ = \Pi^+ \cap \Pi_\nu$ be the system of positive roots of Π_ν. An affine weight λ is ν*-dominant* if $(\lambda : \check{\alpha}) \in \mathbb{N}$ for all $\alpha \in \Pi_\nu^+$. Let w_ν be the longest element in W_ν. We consider the category $\tilde{\mathbf{O}}^\nu$ of the $\mathbf{g}$-modules M such that

- $M = \bigoplus_{\lambda \in \mathbf{t}^*} M_\lambda$ with $M_\lambda = \{m \in M; xm = \lambda(x)m,\, x \in \mathbf{t}\}$,
- $U(\mathbf{p}_\nu)\, m$ is finite dimensional for each $m \in M$.

For a ν-dominant affine weight λ let $V^\nu(\lambda)$, $L(\lambda)$ be the parabolic Verma module with the highest weight λ and its simple top. Let $\mathbf{O}^\nu \subset \tilde{\mathbf{O}}^\nu$ be the full subcategory of the modules such that the highest weight of any of its simple subquotients is of the form

$$\tilde{\lambda}_e = \lambda_e + z_\lambda \delta, \quad \lambda_e = \lambda - (e+m)\Lambda_0, \quad z_\lambda = \langle \lambda : 2\rho + \lambda \rangle / 2e,$$

where $\lambda \in \mathbf{t}^*$ is an integral weight and $e \neq 0$. We abbreviate

$$V^\nu(\lambda_e) = V^\nu(\tilde{\lambda}_e), \quad L(\lambda_e) = L(\tilde{\lambda}_e).$$

For $\mu \in \mathcal{P}$ and $e > 0$ we use the following notation

- $\mathrm{o}_{\mu,-}$ is an antidominant integral classical affine weight of level $-e-m$ whose stabilizer for the $\bullet$-action of $\widehat{W}$ is equal to W_μ,
- $\mathrm{o}_{\mu,+} = -\mathrm{o}_{\mu,-} - 2\hat{\rho}$ is a dominant integral classical affine weight of level $e-m$ whose stabilizer for the $\bullet$-action of $\widehat{W}$ is again equal to W_μ.

Let $\mathbf{O}^\nu_{\mu,\pm}$ be the full subcategory of $\mathbf{O}^\nu$ consisting of the modules such that the highest weight of any of its simple subquotients is linked to $\mathrm{o}_{\mu,\pm}$. Note that $\mathbf{O}^\nu_{\mu,\pm}$ is a direct summand in $\mathbf{O}^\nu$ by [39, thm. 6.1].

2.10. **Truncations.** Fix $e > 0$, $\mu, \nu \in \mathcal{P}$ and $w \in \widehat{W}$. Let $I_\mu^{\min}$, $I_\mu^{\max}$ be the sets of minimal and maximal length representatives of the left cosets in $\widehat{W}/W_\mu$.

- Consider the poset $I_{\mu,-} = (I_\mu^{\max}, \preccurlyeq)$ with $\preccurlyeq$ equal to the Bruhat order $\leqslant$. Let $I^\nu_{\mu,-} = \{x \in I_{\mu,-}\,;\, x \bullet \mathrm{o}_{\mu,-} \text{ is } \nu\text{-dominant}\}$. By Appendix C we have
$$\mathrm{Irr}(\mathbf{O}^\nu_{\mu,-}) = \{L(x \bullet \mathrm{o}_{\mu,-})\,;\, x \in I^\nu_{\mu,-}\},$$
$$I^\nu_{\mu,-} = \{x w_\mu\,;\, x \in (I_\nu^{\max})^{-1} \cap I_\mu^{\min}\}.$$
Let ${}^w\mathbf{O}^\nu_{\mu,-}$ be the category consisting of the finitely generated modules in $\mathbf{O}^\nu_{\mu,-}[{}^wI^\nu_{\mu,-}]$, where ${}^wI^\nu_{\mu,-} = \{x \in I^\nu_{\mu,-}\,;\, x \preccurlyeq w\}$. It is a highest weight category (for the order $\preccurlyeq$) with
$$\mathrm{Irr}({}^w\mathbf{O}^\nu_{\mu,-}) = \{L(x \bullet \mathrm{o}_{\mu,-})\,;\, x \in {}^wI^\nu_{\mu,-}\},$$
$$\Delta({}^w\mathbf{O}^\nu_{\mu,-}) = \{V^\nu(x \bullet \mathrm{o}_{\mu,-})\,;\, x \in {}^wI^\nu_{\mu,-}\}.$$
- Consider the poset $I_{\mu,+} = (I_\mu^{\min}, \preccurlyeq)$ with $\preccurlyeq$ the opposit Bruhat order. Let $I^\nu_{\mu,+} = \{x \in I_{\mu,+}\,;\, x \bullet \mathrm{o}_{\mu,+} \text{ is } \nu\text{-dominant}\}$. By Appendix C we have
$$\mathrm{Irr}(\mathbf{O}^\nu_{\mu,+}) = \{L(x \bullet \mathrm{o}_{\mu,+})\,;\, x \in I^\nu_{\mu,+}\},$$
$$I^\nu_{\mu,+} = \{x w_\mu\,;\, x \in (I_\nu^{\min})^{-1} \cap I_\mu^{\max}\},$$

Let ${}^w\mathbf{O}^\nu_{\mu,+}$ be the category consisting of the finitely generated objects in $\mathbf{O}^\nu_{\mu,+}({}^wI^\nu_{\mu,+})$, where ${}^wI^\nu_{\mu,+} = \{x \in I^\nu_{\mu,+}\,;\, x \succcurlyeq w\}$. By Remark 2.14, it is a highest weight category (for the order $\preccurlyeq$) with

$$\mathrm{Irr}({}^w\mathbf{O}^\nu_{\mu,+}) = \{L(x\bullet \mathrm{o}_{\mu,+})\,;\, x \in {}^wI^\nu_{\mu,+}\},$$
$$\Delta({}^w\mathbf{O}^\nu_{\mu,+}) = \{V^\nu(x\bullet \mathrm{o}_{\mu,+})\,;\, x \in {}^wI^\nu_{\mu,+}\}.$$

We abbreviate ${}^w\mathbf{O}^\nu_{\mu,\pm}$ for either ${}^w\mathbf{O}^\nu_{\mu,+}$ or ${}^w\mathbf{O}^\nu_{\mu,-}$. Let ${}^wP^\nu(x\bullet \mathrm{o}_{\mu,\pm})$ be the projective cover of $L(x\bullet \mathrm{o}_{\mu,\pm})$. If $\nu=\phi$ is regular we abbreviate ${}^w\mathbf{O}_{\mu,\pm} = {}^w\mathbf{O}^\phi_{\mu,\pm}$, and so on, suppressing ϕ in the notation. By Remark 2.12 we have an anti-isomorphism of posets $I^\nu_{\mu,-}\to I^\nu_{\mu,+}$, $x\mapsto x_+ = w_\nu x w_\mu$. Let $x\mapsto x_-$ denote its inverse.

Proposition 2.10. *For $w\in I^\nu_{\mu,-}$ and $v = w_+ \in I^\nu_{\mu,+}$ we have*

(a) the tilting equivalence is an exact contravariant equivalence ${}^w\mathbf{O}^{\nu,\Delta}_{\mu,-}\to {}^v\mathbf{O}^{\nu,\Delta}_{\mu,+}$ which takes $V^\nu(x\bullet \mathrm{o}_{\mu,-})$ to $V^\nu(y\bullet \mathrm{o}_{\mu,+})$ with $y = x_+$,

(b) there is a $\mathbb{C}$-algebra isomorphism $D({}^wR^\nu_{\mu,-}) = {}^vR^\nu_{\mu,+}$ such that $D(1_x) = 1_y$.

Proof. Part (b) is just a reformulation of (a). By [39, thm. 6.6] there is an exact contravariant autoequivalence D of $\mathbf{O}^{\nu,\Delta}$ which takes $V^\nu(\lambda)$ to $V^\nu(-w_\nu(\lambda+\hat\rho)-\hat\rho)$. Note that $-w_\nu(x\bullet \mathrm{o}_{\mu,-}+\hat\rho)-\hat\rho = w_\nu x\bullet \mathrm{o}_{\mu,+}$. Thus the proposition follows from Remarks 2.7, 2.12. □

Remark 2.11. The highest weight category ${}^w\mathbf{O}^\nu_{\mu,\pm}$ does not depend on the choice of $\mathrm{o}_{\mu,\pm}$ and e but only on μ, ν, see [14, thm. 11].

Remark 2.12. (a) Appendix C gives an anti-isomorphism of posets $I^\nu_{\mu,\mp}\to I^\nu_{\mu,\pm}$, $x\mapsto x_\pm$. If $w\in I^\nu_{\mu,\mp}$ and $v = w_\pm$ this anti-isomorphism takes ${}^wI^\nu_{\mu,\mp}$ onto ${}^vI^\nu_{\mu,\pm}$.

(b) We have an isomorphism of posets $I^\nu_{\mu,\pm}\to I^\mu_{\nu,\pm}$, $x\mapsto x^{-1}$. If $w\in I^\nu_{\mu,\pm}$ and $v = w^{-1}$ this isomorphism takes ${}^wI^\nu_{\mu,\pm}$ onto ${}^vI^\mu_{\nu,\pm}$.

(c) We abbreviate $x^{-1}_\pm = (x_\pm)^{-1} = (x^{-1})_\pm$. The assignment $x\mapsto x^{-1}_\pm$ is an anti-isomorphism of posets $I^\nu_{\mu,\mp}\to I^\mu_{\nu,\pm}$.

Remark 2.13. Let i be the inclusion ${}^w\mathbf{O}^\nu_{\mu,\pm}\to {}^w\mathbf{O}_{\mu,\pm}$. The left adjoint functor τ to i takes a module to its maximal quotient which lies in ${}^w\mathbf{O}^\nu_\mu$. We'll call i the *parabolic inclusion functor* and τ the *parabolic truncation functor*. We have

(a) $i(L(x\bullet \mathrm{o}_{\mu,\pm})) = L(x\bullet \mathrm{o}_{\mu,\pm})$ for $x\in {}^wI^\nu_{\mu,\pm}$,
(b) $\tau({}^wP(x\bullet \mathrm{o}_{\mu,\pm})) = {}^wP^\nu(x\bullet \mathrm{o}_{\mu,\pm})$ for $x\in {}^wI^\nu_{\mu,\pm}$,
(c) $\tau({}^wP(x\bullet \mathrm{o}_{\mu,\pm})) = 0$ for $x\in {}^wI_{\mu,\pm}\setminus {}^wI^\nu_{\mu,\pm}$.

The same argument as in [5, thm. 3.5.3], using [18, prop. 3.41], implies that i is injective on extensions in ${}^w\mathbf{O}^\nu_{\mu,-}$. For $w\in I^\nu_{\mu,-}$ and $v = w_+$, the tilting equivalence yields an equivalence of derived categories $\mathbf{D}^b({}^v\mathbf{O}^\nu_{\mu,+})\to \mathbf{D}^b({}^w\mathbf{O}^\nu_{\mu,-})$. Thus i is also injective on extensions in ${}^w\mathbf{O}^\nu_{\mu,+}$.

Remark 2.14. A simple module $L(x\bullet \mathrm{o}_{\mu,+})$ in $\mathbf{O}^\nu_{\mu,+}$ has a projective cover $P^\nu(x\bullet \mathrm{o}_{\mu,+})$, see e.g., [39]. Let $\mathbf{P}\subset \mathbf{O}^\nu_{\mu,+}$ be the full subcategory with set of objects $\{P^\nu(x\bullet \mathrm{o}_{\mu,+})\}$. Consider the algebra without 1 given by

$$Q = \bigoplus_{x,y}\mathrm{End}_{\mathbf{O}^\nu_{\mu,+}}\big(P^\nu(x\bullet \mathrm{o}_{\mu,+}), P^\nu(y\bullet \mathrm{o}_{\mu,+})\big)^{\mathrm{op}},$$

where x, y run over $I^\nu_{\mu,+}$. Set $e = \sum_x 1_x$ in Q, where $x\in {}^wI^\nu_{\mu,+}$. We have

$$eQe = \mathrm{End}_{\mathbf{O}^\nu_{\mu,+}}\big(\bigoplus_x P^\nu(x\bullet \mathrm{o}_{\mu,+})\big)^{\mathrm{op}},\quad x\in {}^wI^\nu_{\mu,+}.$$

By [32, thm. 3.1], assigning to M the restriction of the functor $\mathrm{Hom}_{\mathbf{O}^\nu_{\mu,+}}(\bullet, M)$ to $\mathbf{P}$ yields an equivalence of abelian $\mathbb{C}$-categories $\mathbf{O}^\nu_{\mu,+}\to \mathbf{Mod}(Q)$. Consider the

adjoint pair of functors (F, G) given by

$$F : \mathbf{Mod}(Q) \to \mathbf{Mod}(eQe), \quad M \mapsto eM = \mathrm{Hom}_Q(Qe, M),$$
$$G : \mathbf{Mod}(eQe) \to \mathbf{Mod}(Q), \quad M \mapsto \mathrm{Hom}_{eQe}(eQ, M).$$

The functor F is a *quotient* functor, i.e., we have $F \circ G = \mathbf{1}$, and its kernel is $\mathbf{O}^\nu_{\mu,+}[\not\geq w]$. Therefore, F factors to an equivalence of abelian categories

$${}^w\mathbf{O}^\nu_{\mu,+} \to \mathbf{mod}(eQe).$$

Since eQe is a finite dimensional algebra, the axioms of a highest weight category are now easily verified for ${}^w\mathbf{O}^\nu_{\mu,+}$.

Remark 2.15. Taking a module $M \in \mathbf{O}^\nu_{\mu,\pm}$ to its graded dual $M^* = \bigoplus_{\lambda\in\mathbf{t}^*}(M_\lambda)^*$ with the contragredient $\mathbf{g}$-action yields a duality on $\mathbf{O}^\nu_{\mu,\pm}$ called the BGG duality. Since the BGG duality fixes the simple modules it is easy to see that it factors to a duality on the highest weight category ${}^w\mathbf{O}^\nu_{\mu,\pm}$.

2.11. **The main result.** Fix $e, f > 0$ and $\mu, \nu \in \mathcal{P}$. We choose the integral classical affine weights $\mathrm{o}_{\mu,\pm}$ and $\mathrm{o}_{\nu,\pm}$ such that $\mathrm{o}_{\mu,\pm}$ has level $\pm e - m$ and $\mathrm{o}_{\nu,\pm}$ has level $\pm f - m$. For $w \in \widehat{W}$ let ${}^wT^\nu_{\mu,\pm}$, ${}^wP^\nu_{\mu,\pm}$ and ${}^wL^\nu_{\mu,\pm}$ be the full tilting module, the full projective module and the direct sum of all simple modules in ${}^w\mathbf{O}^\nu_{\mu,\pm}$. For $x \in {}^wI^\nu_{\mu,\pm}$ the projections

$${}^wL^\nu_{\mu,\pm} \to L(x \bullet \mathrm{o}_{\mu,\pm}), \quad {}^wP^\nu_{\mu,\pm} \to {}^wP^\nu(x \bullet \mathrm{o}_{\mu,\pm})$$

define idempotents in the $\mathbb{C}$-algebras

$${}^w\bar{R}^\nu_{\mu,\pm} = \mathrm{Ext}_{{}^w\mathbf{O}^\nu_{\mu,\pm}}({}^wL^\nu_{\mu,\pm}), \quad {}^wR^\nu_{\mu,\pm} = \mathrm{End}_{{}^w\mathbf{O}^\nu_{\mu,\pm}}({}^wP^\nu_{\mu,\pm})^{\mathrm{op}}.$$

Both are denoted by the symbol 1_x.

Theorem 2.16. *Let $w \in I^\nu_{\mu,+}$ and $v = w_-^{-1} \in I^\mu_{\nu,-}$. We have $\mathbb{C}$-algebra isomorphisms ${}^wR^\nu_{\mu,+} = {}^v\bar{R}^\mu_{\nu,-}$ and ${}^w\bar{R}^\nu_{\mu,+} = {}^vR^\mu_{\nu,-}$ such that $1_x \mapsto 1_y$ with $y = x_-^{-1}$. The graded $\mathbb{C}$-algebras ${}^w\bar{R}^\nu_{\mu,+}$ and ${}^v\bar{R}^\mu_{\nu,-}$ are Koszul and are Koszul dual to each other. The categories ${}^w\mathbf{O}^\nu_{\mu,+}$, ${}^v\mathbf{O}^\mu_{\nu,-}$ are Koszul and are Koszul dual to each other.*

Remark 2.17. Indeed, we'll prove that ${}^w\bar{R}^\nu_{\mu,+}$ and ${}^v\bar{R}^\mu_{\nu,-}$ are balanced. In particular, Koszul duality commutes with Ringel duality. Note that we equipped the highest weight categories ${}^v\mathbf{O}^\mu_{\nu,-}$, ${}^w\mathbf{O}^\nu_{\mu,+}$ with the Bruhat and opposite Bruhat order. We compute both the Koszul dual and the Ringel dual with respect to this order, rather than the BGG order, as it is usually done in the literature.

3. Moment graphs, deformed category O and localization

First, we fix some general notation. Fix $e, f > 0$ and $\mu, \nu \in \mathcal{P}$. Let V be a finite dimensional $\mathbb{C}$-vector space. Let S be the symmetric $\mathbb{C}$-algebra over V, with the grading such that V has the degree 2. Let S_0 be the localization of S at the ideal VS. Let k be the residue field of S_0. Note that $k = \mathbb{C}$. By A we'll always denote a commutative, noetherian, integral domain which is a graded S-algebra with 1.

3.1. **Moment graphs.** Let us recall some basic fact on moment graphs. In this section A is the localization of S with respect to some multiplicative subset.

Definition 3.1. *A moment graph over V is a tuple $\mathcal{G} = (I, H, \alpha)$ where (I, H) is a graph with a set of vertices I, a set of edges H, each edge joins two vertices, and α is a map $H \to \mathbb{P}(V)$, $h \mapsto k\alpha_h$.*

Definition 3.2. *An order on $\mathcal{G}$ is a partial order $\preccurlyeq$ on I such that the two vertices joined by an edge are comparable.*

Remark 3.3. We use the terminology in [21]. In [15] a moment graph is ordered. We'll also assume that $\mathcal{G}$ is *finite*, i.e., the sets I and H are finite.

Given an order $\preccurlyeq$ on $\mathcal{G}$ let h', h'' denote the *origin* and the *goal* of the edge h, i.e., the two vertices joined by h with $h' \prec h''$. Let d_x be the set of edges with goal x. Let u_x be the set of edges with origin x. Let $e_x = d_x \bigsqcup u_x$.

Definition 3.4. *A graded A-sheaf over $\mathcal{G}$ is a tuple $\mathcal{M} = (\mathcal{M}_x, \mathcal{M}_h, \rho_{x,h})$ with*

- *a graded A-module $\mathcal{M}_x$ for each $x \in I$,*
- *a graded A-module $\mathcal{M}_h$ for each $h \in H$ such that $\alpha_h \mathcal{M}_h = 0$,*
- *a graded A-module homomorphism $\rho_{x,h} : \mathcal{M}_x \to \mathcal{M}_h$ for h adjacent to x.*

A morphism $\mathcal{M} \to \mathcal{N}$ is a tuple f of graded A-module homomorphisms $f_x : \mathcal{M}_x \to \mathcal{N}_x$ and $f_h : \mathcal{M}_h \to \mathcal{N}_h$ which are compatible with $(\rho_{x,h})$.

For $J \subset I$ we set

$$\mathcal{M}(J) = \big\{(m_x)_{x\in J}\,;\, m_x \in \mathcal{M}_x,\ \rho_{h',h}(m_{h'}) = \rho_{h'',h}(m_{h''})\big\}.$$

The *space of global sections* of the graded A-sheaf $\mathcal{M}$ is the graded A-module $\mathcal{M}(I)$. We say that $\mathcal{M}$ has *finite type* if all $\mathcal{M}_x$ and all $\mathcal{M}_h$ are finitely generated graded A-modules. The graded A-module $\mathcal{M}(I)$ is finitely generated for $\mathcal{M}$ of finite type. The *structural algebra* of $\mathcal{G}$ is the graded A-algebra $\bar{Z}_A$ given by

$$\bar{Z}_A = \big\{(a_x) \in A^{\oplus I}\,;\, a_{h'} - a_{h''} \in \alpha_h A\big\}.$$

Definition 3.5. *Let $\bar{\mathbf{F}}_A$ be the category of the graded A-sheaves of finite type over $\mathcal{G}$ whose stalks are torsion free A-modules. Let $\bar{\mathbf{Z}}_A$ be the category of the graded $\bar{Z}_A$-modules which are finitely generated and torsion free over A.*

We'll call again graded A-sheaves the objects of $\bar{\mathbf{Z}}_A$. The categories $\bar{\mathbf{F}}_A$ and $\bar{\mathbf{Z}}_A$ are Krull-Schmidt graded A-categories (because they are Hom-finite k-categories and each idempotent splits). The global sections functor Γ has a left adjoint $\mathcal{L}$ called the *localization functor* [15, thm. 3.6], [21, prop. 2.14]. We say that $\mathcal{M}$ is *generated by global sections* if the counit $\mathcal{L}\Gamma(\mathcal{M}) \to \mathcal{M}$ is an isomorphism, or, equivalently, if $\mathcal{M}$ belongs to the essential image of $\mathcal{L}$. This implies that the obvious map $\mathcal{M}(\mathcal{G}) \to \mathcal{M}_x$ is surjective for each x. For $E \subset H$ and $J \subset I$ we set

$$\mathcal{M}_E = \bigoplus_{h\in E} \mathcal{M}_h,$$
$$\rho_{J,E} = \bigoplus_{x\in J}\bigoplus_{h\in E} \rho_{x,h} : \mathcal{M}(J) \to \mathcal{M}_E.$$

We abbreviate $\rho_{x,E} = \rho_{\{x\},E}$. Given an order $\preccurlyeq$ on $\mathcal{G}$ we set

$$\mathcal{M}_{\partial x} = \mathrm{Im}(\rho_{\prec x, d_x}), \quad \mathcal{M}^x = \mathrm{Ker}(\rho_{x,e_x}), \quad \mathcal{M}_{[x]} = \mathrm{Ker}(\rho_{x,d_x}).$$

We call $\mathcal{M}_x$ the *stalk* of $\mathcal{M}$ at x, and $\mathcal{M}^x$ its *costalk* at x. Note that $\mathcal{M}^x$, $\mathcal{M}_{[x]}$, $\mathcal{M}_x$ are graded $\bar{Z}_A$-modules such that $\mathcal{M}^x \subset \mathcal{M}_{[x]} \subset \mathcal{M}_x$. Assume that $\mathcal{M}$ is generated by global sections. We say that $\mathcal{M} \in \bar{\mathbf{F}}_A$ is *flabby* if $\mathrm{Im}(\rho_{x,d_x}) = \mathcal{M}_{\partial x}$ for each x, see [21, lem. 3.3], and that it is Δ-*filtered* if it is flabby and if the graded $\bar{Z}_A$-module $\mathcal{M}_{[x]}$ is a free graded A-module for each x, see [15, lem. 4.8].

Definition 3.6. *Let $\bar{\mathbf{F}}^{\Delta}_{A,\preccurlyeq}$ be the full subcategory of $\bar{\mathbf{F}}_A$ consisting of Δ-filtered objects. Let $\bar{\mathbf{Z}}^{\Delta}_{A,\preccurlyeq}$ be the essential image of $\bar{\mathbf{F}}^{\Delta}_{A,\preccurlyeq}$ by Γ. We may abbreviate $\bar{\mathbf{F}}^{\Delta}_A = \bar{\mathbf{F}}^{\Delta}_{A,\preccurlyeq}$ and $\bar{\mathbf{Z}}^{\Delta}_A = \bar{\mathbf{Z}}^{\Delta}_{A,\preccurlyeq}$. They are Krull-Schmidt exact graded A-categories.*

The functors $\mathcal{L}$, Γ are mutually inverse equivalences of graded exact A-categories between $\bar{\mathbf{F}}_A^\Delta$ and $\bar{\mathbf{Z}}_A^\Delta$. For $M \in \bar{\mathbf{Z}}_A$ we set

$$M^x = \mathcal{L}(M)^x, \quad M_{[x]} = \mathcal{L}(M)_{[x]}, \quad M_x = \mathcal{L}(M)_x,$$
$$M_{\prec x} = \mathcal{L}(M)_{\prec x}, \quad M_{\partial x} = \mathcal{L}(M)_{\partial x}, \quad M_h = \mathcal{L}(M)_h.$$

Example 3.7. The stalk of $\bar{Z}_A$ at any vertex x is A.

Let M_J be the image of the obvious map

$$M \to \Gamma\mathcal{L}(M) \to \bigoplus_{x\in J} M_x.$$

It is a graded $(\bar{Z}_A)_J$-module. By [15, lem. 4.5] we have an exact sequence in $\bar{\mathbf{Z}}_A^\Delta$

$$0 \to M' \to M \to M'' \to 0$$

if and only if for each x the following sequence of A-modules is exact

$$0 \to M'_{[x]} \to M_{[x]} \to M''_{[x]} \to 0.$$

Remark 3.8. The canonical functor $\bar{\mathbf{Z}}_A^\Delta \to \mathbf{gmod}(\bar{Z}_A)$ is exact by [16, lem. 2.12].

We say that $M \in \bar{\mathbf{Z}}_A^\Delta$ is *projective* if the functor $\mathrm{Hom}_{\bar{\mathbf{Z}}_A}(M, \bullet)$ maps short exact sequences to short exact sequences. We say that $M \in \bar{\mathbf{Z}}_A$ is *F-projective* if

- $\mathcal{L}(M)$ is flabby,
- M_x is a projective graded A-module for each x,
- $M_h = M_{h'}/\alpha_h M_{h'}$ and $\rho_{h',h}$ is the canonical map for each h,

If $M \in \bar{\mathbf{Z}}_A^\Delta$ is F-projective then it is projective [15, prop. 5.1]. We say that $M \in \bar{\mathbf{Z}}_A^\Delta$ is *tilting* if the contravariant functor $\mathrm{Hom}_{\bar{\mathbf{Z}}_A}(\bullet, M)$ maps short exact sequences to short exact sequences.

Let $M^* = \bigoplus_i (M^*)^i$, with $(M^*)^i = \mathrm{gHom}_A(M, A\{i\})$, be the *graded dual* of a graded A-module M. Here gHom_A is the Hom's space of graded A-modules. Since $\bar{Z}_A$ is commutative, the graded dual of a graded $\bar{Z}_A$-module M is a $\bar{Z}_A$-module. Thus there is a duality $D : \bar{\mathbf{Z}}_A \to \bar{\mathbf{Z}}_A$, $M \mapsto M^*$ which yields an exact contravariant equivalence $\bar{\mathbf{Z}}_{A,\preccurlyeq}^\Delta \to \bar{\mathbf{Z}}_{A,\succcurlyeq}^\Delta$. This duality yields bijections

$$\mathrm{Proj}(\bar{\mathbf{Z}}_{A,\preccurlyeq}^\Delta) \to \mathrm{Tilt}(\bar{\mathbf{Z}}_{A,\succcurlyeq}^\Delta), \quad \mathrm{Tilt}(\bar{\mathbf{Z}}_{A,\preccurlyeq}^\Delta) \to \mathrm{Proj}(\bar{\mathbf{Z}}_{A,\succcurlyeq}^\Delta).$$

We say that $\mathcal{G}$ is a *GKM-graph* if $k\alpha_{h_1} \neq k\alpha_{h_2}$ for edges $h_1 \neq h_2$ adjacent to the same vertex. The *support* of a graded A-sheaf M on $\mathcal{G}$ is the set

$$\mathrm{supp}(M) = \{x \in I\,;\, M_x \neq 0\}.$$

Definition 3.9. *Let $(\mathcal{G}, \preccurlyeq)$ be an ordered GKM-graph. There is a unique object $\bar{B}_{A,\preccurlyeq}(x)$ in $\bar{\mathbf{Z}}_A$ which is indecomposable, F-projective, supported on the coideal $\{\succcurlyeq x\}$ and with $\bar{B}_{A,\preccurlyeq}(x)_x = A$. We call $\bar{B}_{A,\preccurlyeq}(x)$ a BM-sheaf. We may abbreviate*

$$\bar{B}_A(x) = \bar{B}_{A,\preccurlyeq}(x), \quad \bar{C}_A(x) = \bar{C}_{A,\preccurlyeq}(x) = D(\bar{B}_{A,\succcurlyeq}(x)).$$

Remark 3.10. The existence and unicity of BM-sheaves is proved in [15, thm. 5.2] using the Braden-MacPherson algorithm [7, sec. 1.4]. The construction of $\bar{B}_A(x)$ is as follows. We must define $\bar{B}_A(x)_y$, $\bar{B}_A(x)_h$ and $\rho_{y,h}$ for each y, h.

- We set $\bar{B}_A(x)_y = 0$ for $y \not\succcurlyeq x$.
- We set $\bar{B}_A(x)_x = A$.
- Let $y \succ x$ and suppose we have already constructed $\bar{B}_A(x)_z$ and $\bar{B}_A(x)_h$ for any z, h such that $y \succ z, h', h'' \succcurlyeq x$. For $h \in d_y$ let
  - $\bar{B}_A(x)_h = \bar{B}_A(x)_{h'}/\alpha_h \bar{B}_A(x)_{h'}$ and $\rho_{h',h}$ is the canonical map,
  - $\bar{B}_A(x)_{\partial y} = \mathrm{Im}(\rho_{\prec y, d_y}) \subset \bar{B}_A(x)_{d_y}$,
  - $\bar{B}_A(x)_y$ is the projective cover of the graded A-module $\bar{B}_A(x)_{\partial y}$,

– $\rho_{y,h}$ is the composition of the projective cover map $\bar{B}_A(x)_y \to \bar{B}_A(x)_{\partial y}$ with the obvious projection $\bar{B}_A(x)_{d_y} \to \bar{B}_A(x)_h$.

Definition 3.11. *There is a unique object $\bar{V}_A(x)$ in $\bar{\mathbf{Z}}_A$ that is isomorphic to A as a graded A-module and on which the element (z_y) of $\bar{Z}_A$ acts by multiplication with z_x. We call it a Verma-sheaf.*

Proposition 3.12. *(a) The Verma-sheaves are Δ-filtered and self-dual.*
(b) We have $\bar{V}_A(x)^y = \bar{V}_A(x)_y = \bar{V}_A(x)_{[y]} = A$ if $x = y$ and 0 else.
(c) For $M \in \bar{\mathbf{Z}}_A$ we have

$$\operatorname{Hom}_{Z_A}\big(\bar{V}_A(x)\{i\}, M\big) = M^x\{-i\},$$
$$\operatorname{Hom}_{Z_A}\big(M, \bar{V}_A(y)\{j\}\big) = (M_y)^*\{j\},$$
$$\operatorname{Hom}_{\bar{Z}_A}\big(\bar{V}_A(x)\{i\}, \bar{V}_A(y)\{j\}\big) = \delta_{x,y}\, A^{i-j}.$$

(d) A Δ-filtered graded A-sheaf M is an extension of shifted Verma-sheaves.

Remark 3.13. A Δ-filtered graded A-sheaf M has a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ by graded $\bar{Z}_A$-submodules with [15, rk. 4.4, sec. 4.5]

$$\bigoplus_{r=1}^{n} M_r/M_{r-1} = \bigoplus_y M_{[y]}, \quad M_r/M_{r-1} = \bar{V}_A(y_r)\{i_r\}, \quad r \leqslant s \Rightarrow y_s \preccurlyeq y_r.$$

In particular, the graded A-module M is free and finitely generated. If $M \in \bar{\mathbf{Z}}_A^\Delta$ and if B is projective in $\bar{\mathbf{Z}}_A^\Delta$, the graded A-module $\operatorname{Hom}_{Z_A}(B, M)$ has a finite filtration whose associated graded is

$$\bigoplus_{r=1}^{n} \operatorname{Hom}_{Z_A}(B, M_r/M_{r-1}) = \bigoplus_{r=1}^{n} \operatorname{Hom}_A(B_{y_r}, A\{i_r\}).$$

Remark 3.14. By Proposition 3.12, we have $(M_y)^* = (DM)^y$.

Remark 3.15. The category $\bar{\mathbf{Z}}_A^\Delta$ is graded. Forgetting the gradings we get the A-category $\mathbf{Z}_A^\Delta$ and the objects $B_A(x)$, $C_A(x)$.

Remark 3.16. For a morphism of local S-algebras $A \to A'$ we have a *base change* functor $\bullet \otimes_A A' : \bar{\mathbf{Z}}_A \to \bar{\mathbf{Z}}_{A'}$ which takes $\bar{\mathbf{Z}}_A^\Delta$ to $\bar{\mathbf{Z}}_{A'}^\Delta$. If A' is flat as an A-module the canonical map gives an isomorphism [21, sec. 2.7,3.15]

$$A' \operatorname{Hom}_{\bar{\mathbf{Z}}_A}(M, N) \to \operatorname{Hom}_{\bar{\mathbf{Z}}_{A'}}(A'M, A'N).$$

3.2. **The moment graph of O.** In this section we set $V = \mathbf{t}$, $w \in \widehat{W}$, and we assume that A is the localization of S with respect to some multiplicative subset.

Definition 3.17. *Let ${}^w\mathcal{G}_\mu$ denote the moment graph over $\mathbf{t}$ whose set of vertices is ${}^wI_\mu^{\min} = \{x \in I_\mu^{\min}\ ;\ x \leqslant w\}$, with an edge between x, y if and only if there is an affine reflection $s_\alpha \in \widehat{W}$ such that $x \in s_\alpha y W_\mu$, this edge being labelled by $k\,\check{\alpha}$.*

Let ${}^w\bar{Z}_{A,\mu}$ be the structural algebra of ${}^w\mathcal{G}_\mu$ and let ${}^w\bar{\mathbf{Z}}_{A,\mu}$ be the category of graded ${}^w\bar{Z}_{A,\mu}$-modules which are finitely generated and torsion free over A. Let $\bar{V}_{A,\mu}(x)$ be the Verma-sheaf in ${}^w\bar{\mathbf{Z}}_{A,\mu}$ whose stalk at x is A. Let ${}^w\mathcal{G}_{\mu,-}$ be the ordered moment graph $({}^w\mathcal{G}_\mu, \preccurlyeq)$ with $\preccurlyeq$ equal to the Bruhat order. Let ${}^w\mathcal{G}_{\mu,+}$ be the ordered moment graph $({}^w\mathcal{G}_\mu, \preccurlyeq)$ with $\preccurlyeq$ equal to the opposite Bruhat order. We'll write ${}^w\mathcal{G}_{\mu,\pm}$ for either ${}^w\mathcal{G}_{\mu,-}$ or ${}^w\mathcal{G}_{\mu,+}$. We use a similar notation for all objects attached to ${}^w\mathcal{G}_{\mu,\pm}$. For instance ${}^w\bar{\mathbf{Z}}_{A,\mu,\pm}^\Delta$ is the category of Δ-filtered graded A-sheaves on ${}^w\mathcal{G}_{\mu,\pm}$.

Proposition 3.18. *The BM-sheaves on ${}^w\mathcal{G}_{\mu,\pm}$ are Δ-filtered.*

Proof. The BM-sheaves on ${}^w\mathcal{G}_{\mu,-}$ are Δ-filtered by [15, thm. 5.2]. The BM-sheaves on ${}^w\mathcal{G}_{\mu,+}$ are also Δ-filtered. Indeed, note that a graded A-sheaf is Δ-filtered if and only if the underlying non graded A-sheaf is Δ-filtered. Next, by Proposition 3.32(c) the BM-sheaves on ${}^w\mathcal{G}_{\mu,+}$ are Δ-filtered for $A = S_0$. Thus the claim follows by Remark 3.16. □

For $x \in \widehat{W}$ let $l(x)$ be its lengh. By Proposition 3.18 there is a unique Δ-filtered graded A-sheaf ${}^w\bar{B}_{A,\mu,\pm}(x)$ on ${}^w\mathcal{G}_{\mu,\pm}$ which is indecomposable, projective, supported on the ideal $\{\succcurlyeq x\}$ and whose stalk at x is equal to $A\{\pm l(x)\}$. There is also a unique Δ-filtered graded A-sheaf ${}^w\bar{C}_{A,\mu,\pm}(x)$ which is indecomposable, tilting, supported on the ideal $\{\preccurlyeq x\}$ and whose costalk at x is equal to $A\{\pm l(x)\}$. Note that

$$ {}^w\bar{B}_{A,\mu,\mp}(x) = D({}^w\bar{C}_{A,\mu,\pm}(x)). $$

For a future use, we set

$$ {}^w\bar{B}_{A,\mu,\pm} = \bigoplus_x {}^w\bar{B}_{A,\mu,\pm}(x), \quad {}^w\bar{C}_{A,\mu,\pm} = \bigoplus_x {}^w\bar{C}_{A,\mu,\pm}(x). $$

Remark 3.19. The graded A-sheaf ${}^w\bar{C}_{A,\mu,\pm}(x)$ is filtered by Verma-sheaves. The first term in this filtration is the sub-object

$$ \bar{V}_A(x)\{\pm l(x)\} \subset {}^w\bar{C}_{A,\mu,\pm}(x). $$

The other subquotients are of the form $\bar{V}_A(y)\{j\}$ with $y \prec x$ and $j \in \mathbb{Z}$. The graded A-sheaf ${}^w\bar{B}_{A,\mu,\pm}(x)$ is filtered by Verma-sheaves. The top term in this filtration is the quotient object

$$ {}^w\bar{B}_{A,\mu,\pm}(x) \to \bar{V}_A(x)\{\pm l(x)\}. $$

The other subquotients are of the form $\bar{V}_A(y)\{j\}$ with $y \succ x$ and $j \in \mathbb{Z}$.

Proposition 3.20. *We have* $D({}^w\bar{B}_{A,\mu,+}(x)) = {}^w\bar{B}_{A,\mu,+}(x)$.

Proof. If $\mu = \phi$ is regular then the claim is [16, thm. 6.1]. Assume now that μ is no longer regular. We abbreviate $\bar{B}_\mu(x) = {}^w\bar{B}_{A,\mu,+}(x)$. The regular case implies that $D(\bar{B}_\phi(x)) = \bar{B}_\phi(x)$. We can assume that $w \in I_\mu^{\max}$. Recall that $x \in {}^wI_\mu$. Then Proposition 3.40 yields

$$ \bigoplus_{y \in W_\mu} D(\bar{B}_\mu(x))\{2l(y) - l(w_\mu)\} \oplus D(M) = \bigoplus_{y \in W_\mu} \bar{B}_\mu(x)\{l(w_\mu) - 2l(y)\} \oplus M, $$

where M is a direct sum of objects of the form $\bar{B}_\mu(z)\{j\}$ with $z < x$. We have also $D(\bar{B}_\mu(e)) = \bar{B}_\mu(e)$. By induction we may assume that $D(\bar{B}_\mu(z)) = \bar{B}_\mu(z)$ for all $z < x$. Therefore, we have $D(\bar{B}_\mu(x)) = \bar{B}_\mu(x)$. □

Remark 3.21. The graded sheaves ${}^w\bar{B}_{A,\mu,-}(x)$ are not self-dual.

Proposition 3.22. *The category* ${}^w\bar{\mathbf{Z}}^\Delta_{A,\mu,\pm}$ *is Krull-Schmidt. A projective object is a direct sum of objects of the form* ${}^w\bar{B}_{A,\mu,\pm}(x)\{j\}$. *A tilting object is a direct sum of objects of the form* ${}^w\bar{C}_{A,\mu,\pm}(x)\{j\}$.

Proof. The first claim is obvious by the discussion in the previous section. The third claim follows from the second one via the duality. Now, let P be a projective object. Fix a filtration of P by Verma-sheaves as in Remark 3.13 such that the top Verma-sheaf in this filtration is of the form $\bar{V}_{A,\mu}(x)\{j\}$ with x minimal in $\mathrm{supp}(P)$. Thus there is an epimorphism $P \to \bar{V}_{A,\mu}(x)\{j\}$. There is also an epimorphism ${}^w\bar{B}_{A,\mu,\pm}(x)\{j \mp l(x)\} \to \bar{V}_{A,\mu}(x)\{j\}$. Thus ${}^w\bar{B}_{A,\mu,\pm}(x)$ is a direct summand in P and the proposition follows by induction. □

Let $P^{\mu,q}_{y,x}(t) = \sum_i P^{\mu,q}_{y,x,i}\, t^i$ be Deodhar's parabolic Kazhdan-Lusztig polynomial "of type q" associated with the parabolic subgroup W_μ of $\widehat{W}$ and let $Q^{\mu,-1}_{x,y}(t) = \sum_i Q^{\mu,-1}_{x,y,i}\, t^i$ be Deodhar's inverse parabolic Kazhdan-Lusztig polynomial "of type -1". We use the notation in [24, rk. 2.1]. It is not the usual one. We abbreviate $P_{y,x} = P^{\phi,q}_{y,x}$ and $Q_{y,x} = Q^{\phi,-1}_{y,x}$.

Proposition 3.23. *We have graded A-module isomorphisms*
(a) ${}^w\bar{B}_{A,\mu,+}(x)_y = \bigoplus_{i\geqslant 0} A\{l(x)-2i\}^{\oplus P^{\mu,q}_{y,x,i}}$,
(b) ${}^w\bar{B}_{A,\mu,+}(x)_{[y]} = \bigoplus_{i\geqslant 0} A\{2l(y)-l(x)+2i\}^{\oplus P^{\mu,q}_{y,x,i}}$.

Proof. Part (a) follows from Proposition 3.44 and [24, thm. 1.4]. Part (b) follows from (a) and Proposition 3.20 as in [16, prop. 7.1] (where it is proved for μ regular). More precisely, we'll abbreviate

$$\bar{V}(x) = \bar{V}_{S,\phi}(x), \quad \bar{B}(x) = {}^w\bar{B}_{A,\mu,+}(x), \quad Z = {}^wZ_{S,\mu}.$$

Therefore, we have

$$\bar{B}(x)^y \subset \bar{B}(x)_{[y]} = \mathrm{Ker}(\rho_{y,d_y}) \subset \bar{B}(x)_y.$$

In particular, for $\alpha_{u_y} = \prod_{h\in u_y}\alpha_h$, we have

$$\alpha_{u_y}\,\bar{B}(x)_{[y]} \subset \bar{B}(x)^y.$$

The graded S-module $\bar{B}(x)_y$ is free and $\bar{B}(x)_h = \bar{B}(x)_y/\alpha_h\bar{B}(x)_y$ for $h\in u_y$ because $\bar{B}(x)$ is F-projective. Thus we have

$$\bar{B}(x)^y \subset \alpha_{u_y}\,\bar{B}(x)_y,$$

because α_{h_1} is prime to α_{h_2} in S if $h_1\neq h_2$. We claim that

$$\bar{B}(x)^y = \alpha_{u_y}\,\bar{B}(x)_{[y]}. \tag{3.1}$$

Let $b\in\bar{B}(x)^y$. Write $b=\alpha_{u_y}\,b'$ with $b'\in\bar{B}(x)_y$. If $\rho_{y,h}(b')\neq 0$ with $h\in d_y$ then

$$\rho_{y,h}(b) = \alpha_{u_y}\,\rho_{y,h}(b')\neq 0,$$

because the α_{u_y}-torsion submodule of $\bar{B}(x)_h$ is zero. Indeed, we have $\bar{B}(x)_h = \bar{B}(x)_{h'}/\alpha_h\bar{B}(x)_{h'}$, the S-module $\bar{B}(x)_{h'}$ is free, and α_h is prime to α_{u_y}. This implies the claim (3.1). In particular, we have

$$\bar{B}(x)_{[y]} = \bar{B}(x)^y\{2l(y)\}$$

because $\sharp u_y = l(y)$. Hence, by Remark 3.14 and Proposition 3.20 we have a graded S-module isomorphism

$$(\bar{B}(x)_y)^*\{2l(y)\} = \bar{B}(x)^y\{2l(y)\} = \bar{B}(x)_{[y]}.$$

□

Definition 3.24. *We define the graded k-algebra*

$${}^w\bar{A}_{\mu,\mp} = k\,\mathrm{End}_{{}^wZ_{S_0,\mu}}\big({}^w\bar{B}_{S_0,\mu,\mp}\big)^{\mathrm{op}} = k\,\mathrm{End}_{{}^wZ_{S_0,\mu}}\big({}^w\bar{C}_{S_0,\mu,\pm}\big).$$

Let ${}^wA_{\mu,\pm}$ be the k-algebra underlying ${}^w\bar{A}_{\mu,\pm}$. Let 1_x be the idempotent of ${}^w\bar{A}_{\mu,\pm}$ associated with the direct summand ${}^w\bar{B}_{S_0,\mu,\pm}(x)$ of ${}^w\bar{B}_{S_0,\mu,\pm}$.

Proposition 3.25. *The graded k-algebra ${}^w\bar{A}_{\mu,\pm}$ is basic. Its Hilbert polynomial is*

$$P({}^w\bar{A}_{\mu,+},t)_{x,x'} = \sum_{y\leqslant x,x'} P^{\mu,q}_{y,x}(t^{-2})\,P^{\mu,q}_{y,x'}(t^{-2})\,t^{l(x)+l(x')-2l(y)},$$

$$P({}^w\bar{A}_{\mu,-},t)_{x,x'} = \sum_{y\geqslant x,x'} Q^{\mu,-1}_{x,y}(t^{-2})\,Q^{\mu,-1}_{x',y}(t^{-2})\,t^{2l(y)-l(x)-l(x')}.$$

If $\mu = \phi$ then the following matrix equation holds

$$P({}^w\bar{A}_{\phi,-}, t)\, P({}^v\bar{A}_{\phi,+}, -t) = 1,$$

where $v = w^{-1}$ and the sets of indices of the matrices in the left and right factors are identified through the map $x \mapsto x^{-1}$.

Proof. By Remark 3.13, for each x, x' there is a finite filtration of the graded S_0-module $\mathrm{Hom}_{{}^wZ_{S_0,\mu}}\big({}^w\bar{B}_{S_0,\mu,+}(x'), {}^w\bar{B}_{S_0,\mu,+}(x)\big)$ whose associated graded is

$$\begin{aligned} &= \bigoplus_y \bigoplus_{i\geqslant 0} \mathrm{Hom}_{S_0}\big({}^w\bar{B}_{S_0,\mu,+}(x')_y, S_0\big)\{2l(y) - l(x) + 2i\}^{\oplus P^{\mu,q}_{y,x,i}} \\ &= \bigoplus_y \bigoplus_{i,i'\geqslant 0} S_0\{2l(y) - l(x) - l(x') + 2i + 2i'\}^{\oplus P^{\mu,q}_{y,x,i}\, P^{\mu,q}_{y,x',i'}}. \end{aligned}$$

Thus we have a graded k-vector space isomorphism

$$1_x\, {}^w\bar{A}_{\mu,+} 1_{x'} = \bigoplus_{y\leqslant x,x'} \bigoplus_{i,i'\geqslant 0} k\{2l(y) - l(x) - l(x') + 2i + 2i'\}^{\oplus P^{\mu,q}_{y,x,i}\, P^{\mu,q}_{y,x',i'}},$$

where y, x, x' run over ${}^wI_{\mu,+}$. Therefore, we have

$$P({}^w\bar{A}_{\mu,+}, t)_{x,x'} = \sum_{y\leqslant x,x'} P^{\mu,q}_{y,x}(t^{-2})\, P^{\mu,q}_{y,x'}(t^{-2})\, t^{l(x)+l(x')-2l(y)}.$$

In other words, the following matrix equation holds

$$P({}^w\bar{A}_{\mu,+}, t) = P_{\mu,+}(t)^T\, P_{\mu,+}(t), \quad P_{\mu,+}(t)_{y,x} = P^{\mu,q}_{y,x}(t^{-2})\, t^{l(x)-l(y)}, \quad x, y \in {}^wI_{\mu,+}.$$

Note that $P^{\mu,q}_{y,x,i} = 0$ if $l(y) - l(x) + 2i > 0$ and that if $l(y) - l(x) + 2i = 0$ then $P^{\mu,q}_{y,x,i} = 0$ unless $y = x$. Thus we have

$$P({}^w\bar{A}_{\mu,+}, t)_{x,x'} \in \delta_{x,x'} + t\mathbb{N}[[t]].$$

Hence the graded k-algebra ${}^w\bar{A}_{\mu,+}$ is basic. The matrix equation

$$P({}^w\bar{A}_{\mu,-}, t) = Q_{\mu,-}(t)\, Q_{\mu,-}(t)^T, \quad Q_{\mu,-}(t)_{x,y} = Q^{\mu,-1}_{x,y}(t^{-2})\, t^{l(y)-l(x)}, \quad x, y \in {}^wI_{\mu,+}$$

is proved in Proposition A.5. Hence ${}^w\bar{A}_{\mu,-}$ is also basic. Next, we have, see e.g., [24, (2.39)]

$$\sum_{x\leqslant y\leqslant x'} (-1)^{l(y)-l(x)} Q^{\mu,a}_{x,y}(t)\, P^{\mu,a}_{y,x'}(t) = \delta_{x,x'}, \quad x, x', y \in {}^wI_{\mu,+}, \quad a = -1, q.$$

Further, if $\mu = \phi$ is regular then we have also $P_{y,x}(t) = P^{\mu,q}_{y,x}(t)$ and $Q_{x,y}(t) = Q^{\mu,q}_{x,y}(t)$. So we have the matrix equation

$$Q_{\phi,-}(t)\, P_{\phi,+}(-t) = P_{\phi,+}(-t)\, Q_{\phi,-}(t) = 1.$$

Therefore, we have

$$P({}^w\bar{A}_{\phi,-}, t) P({}^w\bar{A}_{\phi,+}, -t) = Q_{\phi,-}(t)\, Q_{\phi,-}(t)^T\, P_{\phi,+}(-t)^T\, P_{\phi,+}(-t) = 1.$$

The matrix equation in the proposition follows easily, using the fact that $P_{y,x} = P_{y^{-1},x^{-1}}$ and $Q_{x,y} = Q_{x^{-1},y^{-1}}$. □

Remark 3.26. The Hilbert polynomial $P({}^w\bar{A}_{\mu,+}, t)$ can also be computed in the same way as the Hilbert polynomial $P({}^w\bar{A}_{\mu,-}, t)$ in Proposition A.5. The proof above is of independent interest.

Remark 3.27. Forgetting the gradings we define in the same way ${}^wB_{A,\mu,\pm}(x)$, ${}^wB_{A,\mu,\pm}$, ${}^wC_{A,\mu,\pm}(x)$, ${}^wC_{A,\mu,\pm}$, and $V_{A,\mu}(x)$.

Remark 3.28. For $V = \mathbf{t}^*$, let ${}^w\mathcal{G}_\mu^\vee$ be the moment graph over V whose set of vertices is ${}^wI_{\mu,+}$, with an edge between x, y if and only if there is an affine reflection s_α such that $x \in s_\alpha y W_\mu$, this edge being labelled by $k\,\alpha$. Let S be the symmetric k-algebra over V and let A be a commutative, noetherian, integral domain which is a graded S-algebra with 1. We define ${}^w\bar{Z}_{A,\mu}^\vee$, ${}^w\bar{\mathbf{Z}}_{A,\mu}^\vee$, ${}^w\bar{B}_{A,\mu,\pm}(x)^\vee$, ${}^w\bar{C}_{A,\mu,\pm}(x)^\vee$, etc., in the obvious way. Next, set ${}^w\bar{Z}_\mu^\vee = k{}^w\bar{Z}_{S,\mu}^\vee$. Let ${}^w\bar{\mathbf{Z}}_\mu^\vee$ be the category of the finite dimensional graded ${}^w\bar{Z}_\mu^\vee$-modules. We define

$${}^w\bar{B}_{\mu,\pm}^\vee(x) = k{}^w\bar{B}_{S,\mu,\pm}^\vee(x), \quad {}^w\bar{C}_{\mu,\pm}^\vee(x) = k{}^w\bar{C}_{S,\mu,\pm}^\vee(x).$$

Remark 3.29. The results in this section have obvious analogues in finite type. Then, we may omit the truncation and Proposition 3.23, Corollary A.4 yield

$$\bar{B}_{A,\mu,+}(x)_y = \bigoplus_{i\geqslant 0} A\{l(x)-2i\}^{\oplus P_{y,x,i}^{\mu,q}}$$

$$\bar{B}_{A,\mu,-}(x)_y\{l(w_0)\} = \bigoplus_{i\geqslant 0} A\{l(w_0x)-2i\}^{\oplus Q_{x,y,i}^{\mu,-1}},$$

where w_0 is the longuest element in the Weyl group W. Indeed, we have

$$\omega^*\bar{B}_{A,\mu,+}(w_0xw_\mu) = \bar{B}_{A,\mu,-}(x)\{l(w_0w_\mu)\},$$

where $\omega : \mathcal{G}_{\mu,-} \to \mathcal{G}_{\mu,+}$ is the ordered moment graph isomorphism induced by the bijections $I_{\mu,+} \to I_{\mu,+}$, $x \mapsto w_0xw_\mu$ and $\mathbf{t} \to \mathbf{t}$, $h \mapsto w_0(h)$. Note that [24, prop. 2.4,2.6] and Kazhdan-Lusztig's inversion formula give

$$Q_{x,y}^{\mu,-1} = P_{w_0yw_\mu,w_0xw_\mu}^{\mu,q}.$$

3.3. **Deformed category O.** In this section we set $V = \mathbf{t}$ and we assume that A is a local S_0-algebra. Let $w \in \widehat{W}$. For $\lambda \in \mathbf{t}^*$ let A_λ be the $(\mathbf{t}, A)$-bimodule which is free of rank one over A and such that $x \in \mathbf{t}$ acts by multiplication by the image of the element $\lambda(x) + x$ by the canonical map $S_0 \to A$. The *deformed Verma module* with highest weight λ is the $(\mathbf{g}, A)$-bimodule given by

$$V_A(\lambda) = U(\mathbf{g}) \otimes_{U(\mathbf{b})} A_\lambda.$$

We'll write $r \cdot m$ for the action of an element $r \in A$ on an element $m \in M$. The category $\mathbf{O}_{A,\mu,\pm}$ consists of $(\mathbf{g}, A)$-bimodules M such that

- $M = \bigoplus_{\lambda\in\mathbf{t}^*} M_\lambda$ with $M_\lambda = \{m \in M\,;\, xm = (\lambda(x)+x)\cdot m,\ x \in \mathbf{t}\}$,
- $U(\mathbf{b})\,(A\cdot m)$ is finitely generated over A for each $m \in M$,
- the highest weight of any simple subquotient is linked to $\mathrm{o}_{\mu,\pm}$.

The morphisms are the $(\mathbf{g}, A)$-bimodule homomorphisms. We are interested by the following categories :

- ${}^w\mathbf{O}_{A,\mu,-}$ is the thick subcategory of $\mathbf{O}_{A,\mu,-}$ of the finitely generated modules such that the highest weight of any simple subquotient is of the form $\lambda = x \bullet \mathrm{o}_{\mu,-}$ with $x \preccurlyeq w$ and $x \in I_{\mu,-}$.
- ${}^w\mathbf{O}_{A,\mu,+}$ is the category of the finitely generated objects in the Serre quotient of $\mathbf{O}_{A,\mu,+}$ by the thick subcategory of the modules such that the highest weight of any simple subquotient is of the form $\lambda = x \bullet \mathrm{o}_{\mu,+}$ with $x \not\succcurlyeq w$ and $x \in I_{\mu,+}$. We use the same notation for a module in $\mathbf{O}_{A,\mu,+}$ and a module in the quotient category ${}^w\mathbf{O}_{A,\mu,+}$.

Proposition 3.30. *(a) The category ${}^w\mathbf{O}_{A,\mu,\pm}$ is a highest weight category over A. The standard objects are the deformed Verma modules $V_A(x\bullet \mathrm{o}_{\mu,\pm})$ with $x \in {}^wI_{\mu,\pm}$.*

(b) For $w \in I_{\mu,\pm}^\nu$ the tilting equivalence gives a duality $D : {}^w\mathbf{O}_{A,\mu,\pm} \to {}^v\mathbf{O}_{A,\mu,\mp}$ where $v = w_\mp$.

Proof. First, we consider the category ${}^w\mathbf{O}_{A,\mu,-}$. The deformed Verma modules are split, i.e., their endomorphism ring is A. Further, the A-category ${}^w\mathbf{O}_{A,\mu,-}$ is Hom finite. Thus we must check that ${}^w\mathbf{O}_{A,\mu,-}$ has a projective generator and that the projective modules are Δ-filtered. Both statements follow from [13, thm. 2.7].

Next, we consider the category ${}^w\mathbf{O}_{A,\mu,+}$. Once again it is enough to check that ${}^w\mathbf{O}_{A,\mu,+}$ has a projective generator and that the projective modules are Δ-filtered. By [13, thm. 2.7], a simple module $L(x \bullet \mathrm{o}_{\mu,+})$ in $\mathbf{O}_{A,\mu,+}$ has a projective cover $P_A(x \bullet \mathrm{o}_{\mu,+})$. Note that the deformed category O in loc. cit. is indeed a subcategory of $\mathbf{O}_{A,\mu,+}$ containing all finitely generated modules. Since $P_A(x \bullet \mathrm{o}_{\mu,+})$ is finitely generated, the functor $\mathrm{Hom}_{\mathbf{O}_{A,\mu,+}}(P_A(x \bullet \mathrm{o}_{\mu,+}), \bullet)$ commutes with direct limits. Thus, since any module in $\mathbf{O}_{A,\mu,+}$ is the direct limit of its finitely generated submodules, the module $P_A(x \bullet \mathrm{o}_{\mu,+})$ is again projective in $\mathbf{O}_{A,\mu,+}$. Now, the same argument as in Remark 2.14 using [32, thm. 3.1] shows that the functor

$$\mathrm{Hom}_{\mathbf{O}_{A,\mu,+}}\big(\bigoplus_x P_A(x \bullet \mathrm{o}_{\mu,+}), \bullet\big), \quad x \in {}^wI_{\mu,+},$$

factors to an equivalence of abelian A-categories

$${}^w\mathbf{O}_{A,\mu,+} \to \mathbf{mod}(R), \quad R = \mathrm{End}_{\mathbf{O}_{A,\mu,+}}\big(\bigoplus_x P_A(x \bullet \mathrm{o}_{\mu,+})\big)^{\mathrm{op}},$$

where R is a finite projective A-algebra. Using this, the axioms of a highest weight category over A are easily verified for ${}^w\mathbf{O}_{A,\mu,+}$, using, e.g., [36, thm. 4.15]. This finishes the proof of (a).

The tilting equivalence D is constructed in the more general context of highest weight categories over a ring in [36], see Section 2.5 for details. To prove (b) it is enough to check that $D({}^w\mathbf{O}_{A,\mu,+}) = {}^w\mathbf{O}_{A,\mu,-}$. This follows from [14, sec. 2.6] and the characterization of tilting objects in a highest weight category over a ring in [36, lem. 4.21, def. 4.25]. $\square$

Recall that ${}^w\mathbf{O}^{\Delta}_{A,\mu,\pm}$ is the full subcategory of ${}^w\mathbf{O}_{A,\mu,\pm}$ of the Δ-filtered modules. Let ${}^wP_A(x \bullet \mathrm{o}_{\mu,\pm})$ be the projective cover of $V_A(x \bullet \mathrm{o}_{\mu,\pm})$. Set

$${}^wT_A(x \bullet \mathrm{o}_{\mu,\pm}) = D\big({}^wP_A(x \bullet \mathrm{o}_{\mu,\mp})\big).$$

It is a tilting object. For a morphism of local S_0-algebras $A \to A'$ we have an exact *base change* functor ${}^w\mathbf{O}_{A,\mu,\pm} \to {}^w\mathbf{O}_{A',\mu,\pm}$, $M \mapsto A'M = M \otimes_A A'$.

Proposition 3.31. *(a) For $x \in {}^wI_{\mu,\pm}$ we have*

$$k{}^wP_A(x \bullet \mathrm{o}_{\mu,\pm}) = {}^wP(x \bullet \mathrm{o}_{\mu,\pm}), \quad k{}^wT_A(x \bullet \mathrm{o}_{\mu,\pm}) = {}^wT(x \bullet \mathrm{o}_{\mu,\pm}).$$

(b) Projective objects in ${}^w\mathbf{O}^{\Delta}_{A,\mu,\pm}$ are finite direct sums of ${}^wP_A(\lambda)$'s, tilting objects are finite direct sums of ${}^wT_A(\lambda)$'s.

(c) Base change takes projectives to projectives and tiltings to tiltings. The obvious map $A'\,\mathrm{Hom}(M,N) \to \mathrm{Hom}(A'M, A'N)$ is invertible for $M, N \in {}^w\mathbf{O}^{\Delta}_{A,\mu,\pm}$.

Proof. The proposition follows from Proposition 3.30 and from the general theory of a highest weight category over a ring. For instance, the last claim in (c) follows from the proof of [36, prop. 4.30]. See also [13, thm. 2.7, prop. 2.4]. $\square$

Proposition 3.32. *There is a functor $\mathbb{V} : {}^w\mathbf{O}^{\Delta}_{A,\mu,\pm} \to {}^w\mathbf{Z}_{A,\mu}$ such that*

(a) $\mathbb{V}$ is exact (for the exact structure on $\mathbf{mod}({}^wZ_{A,\mu})$),

(b) $\mathbb{V}$ commutes with base change,

(c) if $A = S_0$ and $x \in {}^wI^{\min}_{\mu}$ then we have

- $\mathbb{V} \circ D = D \circ \mathbb{V}$,
- $\mathbb{V}$ *is an equivalence of exact categories* ${}^w\mathbf{O}^{\Delta}_{A,\mu,\pm} \to {}^w\mathbf{Z}^{\Delta}_{A,\mu,\pm}$,
- $\mathbb{V}({}^wP_{S_0}(x \bullet \mathrm{o}_{\mu,\pm})) = {}^wB_{S_0,\mu,\pm}(x)$,

- $\mathbb{V}({}^wT_{S_0}(x\bullet \mathrm{o}_{\mu,\pm})) = {}^wC_{S_0,\mu,\pm}(x)$,
- $\mathbb{V}(V_{S_0}(x\bullet \mathrm{o}_{\mu,\pm})) = V_{S_0,\mu,\pm}(x)$,

(d) if $A = k$ then $\mathbb{V}$ is fully-faithful on projectives in ${}^w\mathbf{O}^{\Delta}_{\mu,+}$.

Proof. A functor $\mathbb{V}$ on $\mathbf{O}^{\Delta}_{A,\mu,\pm}$ is defined in [14]. It is exact for the exact structure on ${}^w\mathbf{Z}_{A,\mu}$ induced by the embedding ${}^w\mathbf{Z}_{A,\mu} \subset \mathbf{mod}({}^wZ_{A,\mu})$ and by the usual exact structure on the right hand side, see [14, prop. 2]. Further, the functor $\mathbb{V}$ commutes with base change and with the tilting equivalence, see [14, prop. 2, sec. before rem. 6]. Here we define $\mathbb{V}$ on ${}^w\mathbf{O}^{\Delta}_{A,\mu,-}$ as the composition of $\mathbb{V}$ and of the embedding ${}^w\mathbf{O}^{\Delta}_{A,\mu,-} \subset \mathbf{O}_{A,\mu,-}$, and we define $\mathbb{V}$ on ${}^w\mathbf{O}^{\Delta}_{A,\mu,+}$ as the composition of $\mathbb{V}$ and D. This proves $(a),(b)$ and the first claim of (c). The second claim of (c) follows from [15, thm. 7.1]. Note that we use here the non-standard exact structure on ${}^w\mathbf{Z}^{\Delta}_{A,\mu,\pm}$, see [15, sec. 7.1] and Remark 3.8. The third claim of (c) follows from [15, prop. 7.2]. The last one follows from the proof of [14, prop. 2, sec. before rem. 6]. Finally, the fourth claim of (c) follows from the second one. Part (d) follows from [40, thm. 3.9]. □

Corollary 3.33. *We have a k-algebra isomorphism ${}^wR_{\mu,\pm} \to {}^wA_{\mu,\pm}$ such that $1_x \mapsto 1_x$ if $x \in {}^wI_{\mu,+}$, and $1_x \mapsto 1_{xw_\mu}$ if $x \in {}^wI_{\mu,-}$.*

Proof. Propositions 3.31, 3.32(c) yield

$$\begin{aligned}{}^wR_{\mu,\pm} &= \mathrm{End}_{{}^w\mathbf{O}_{\mu,\pm}}({}^wP_{\mu,\pm})^{\mathrm{op}}\\ &= k\,\mathrm{End}_{{}^w\mathbf{O}_{S_0,\mu,\pm}}({}^wP_{S_0,\mu,\pm})^{\mathrm{op}}\\ &= k\,\mathrm{End}_{{}^wZ_{S_0,\mu}}({}^wB_{S_0,\mu,\pm})^{\mathrm{op}}\\ &= {}^wA_{\mu,\pm}.\end{aligned}$$

□

Remark 3.34. The highest weight category ${}^w\mathbf{O}^{\nu}_{A,\mu,\pm}$ over A does not depend on the choice of $\mathrm{o}_{\mu,\pm}$ and e but only on μ, ν, see [14, thm. 11].

Remark 3.35. We set ${}^w\bar{Z}_\mu = k{}^w\bar{Z}_{S_0,\mu}$ and ${}^w\bar{B}_{\mu,+} = k{}^w\bar{B}_{S_0,\mu,+}$, compare Remark 3.28. By Proposition 3.32(c) we have

$$\begin{aligned}\mathrm{End}_{{}^wZ_\mu}({}^wB_{\mu,+})^{\mathrm{op}} &= \mathrm{End}_{{}^w\mathbf{O}_{\mu,+}}({}^wP_{\mu,+})^{\mathrm{op}}\\ &= k\,\mathrm{End}_{{}^w\mathbf{O}_{S_0,\mu,+}}({}^wP_{S_0,\mu,+})^{\mathrm{op}}\\ &= k\,\mathrm{End}_{{}^wZ_{S_0,\mu}}({}^wB_{S_0,\mu,+})^{\mathrm{op}}\\ &= {}^wA_{\mu,+}.\end{aligned}$$

Hence we have also a graded k-algebra isomorphism

$$\mathrm{End}_{{}^wZ_\mu}({}^w\bar{B}_{\mu,+})^{\mathrm{op}} = {}^w\bar{A}_{\mu,+}.$$

This isomorphism can also be recovered from the results in Section 3.6.

Remark 3.36. By Corollary A.6 and Proposition 3.32(c) we have

$$\begin{aligned}\mathrm{End}_{{}^wZ_\mu}(\mathbb{V}({}^wP_{\mu,-}))^{\mathrm{op}} &= \mathrm{End}_{{}^wZ_\mu}({}^wB_{\mu,-})^{\mathrm{op}}\\ &= {}^wA_{\mu,-}\\ &= k\,\mathrm{End}_{{}^w\mathbf{O}_{S_0,\mu,-}}({}^wP_{S_0,\mu,-})^{\mathrm{op}}\\ &= \mathrm{End}_{{}^w\mathbf{O}_{\mu,-}}({}^wP_{\mu,-})^{\mathrm{op}}.\end{aligned}$$

This implies that for $A = k$ the functor $\mathbb{V}$ is fully-faithful on projectives in ${}^w\mathbf{O}^{\Delta}_{\mu,-}$. This is an analogue of Proposition 3.32(d) for negative level.

3.4. **Translation functors in O.** In this section we set $V = \mathbf{t}$ and we assume that A is a local S_0-algebra. Assume that $e \neq d \mp m$ and that $o_{\mu,-}$, $o_{\phi,-}$ have the level $-e-m$, $-d-m$ respectively. Then $o_{\mu,\pm} - o_{\phi,\pm}$ is an integral affine weight of level $\pm(e-d)$. It is positive if $\pm(e-d) > -m$ and negative else, see Section 2.8.

Proposition 3.37. *Let $z \in I_\mu^{\max}$, $w \in zW_\mu$ and $\pm(e-d) > -m$. We have k-linear functors*

$$T_{\phi,\mu} : {}^z\mathbf{O}^\Delta_{A,\phi,\pm} \to {}^w\mathbf{O}^\Delta_{A,\mu,\pm}, \quad T_{\mu,\phi} : {}^w\mathbf{O}^\Delta_{A,\mu,\pm} \to {}^z\mathbf{O}^\Delta_{A,\phi,\pm} \tag{3.2}$$

such that the following hold

(a) $T_{\phi,\mu}$, $T_{\mu,\phi}$ are exact,

(b) $T_{\phi,\mu}$, $T_{\mu,\phi}$ are bi-adjoint if $A = S_0$ or k,

(c) $T_{\phi,\mu}$, $T_{\mu,\phi}$ commute with base change.

Proof. The functors $T_{\phi,\mu}$, $T_{\mu,\phi}$ on $\mathbf{O}^\Delta_{A,\phi,\pm}$, $\mathbf{O}^\Delta_{A,\mu,\pm}$ are constructed in [13]. Since $z \in I_\mu^{\max}$, by [14, thm. 4(2)] the functors $T_{\phi,\mu}$, $T_{\mu,\phi}$ preserve the subcategories ${}^z\mathbf{O}^\Delta_{A,\phi,-}$, ${}^w\mathbf{O}^\Delta_{A,\mu,-}$. For the same reason the functors $T_{\phi,\mu}$, $T_{\mu,\phi}$ factor to the categories ${}^z\mathbf{O}^\Delta_{A,\phi,+}$, ${}^w\mathbf{O}^\Delta_{A,\mu,+}$. $\square$

Proposition 3.38. *Let $A = k$, $z \in I_\mu^{\max}$, $w \in zW_\mu$ and $\pm(e-d) > -m$. We have a k-linear functor*

$$T_{\phi,\mu} : {}^z\mathbf{O}_{\phi,\pm} \to {}^w\mathbf{O}_{\mu,\pm} \tag{3.3}$$

such that (3.2), (3.3) *coincide on ${}^z\mathbf{O}^\Delta_{\phi,\pm}$ and the following hold*

(a) $T_{\phi,\mu}$ has a left adjoint functor $T_{\mu,\phi}$,

(b) $T_{\phi,\mu}$ is exact and takes projectives to projectives,

(c) $T_{\phi,\mu}$, $T_{\mu,\phi}$ preserve the parabolic category O and commute with i, τ,

(d) $T_{\mu,\phi}({}^zP^\nu(x \bullet o_{\mu,\pm})) = {}^wP^\nu(xw_\mu \bullet o_{\phi,\pm})$ for $x \in {}^zI_{\mu,\pm}$,

(e) $T_{\phi,\mu}(L(xw_\mu \bullet o_{\phi,\pm})) = L(x \bullet o_{\mu,\pm})$ for $x \in {}^zI_{\mu,\pm}$,

(f) $T_{\phi,\mu}(L(xw_\mu \bullet o_{\phi,\pm})) = 0$ iff $x \in {}^zI_{\phi,\pm} \setminus {}^zI_{\mu,\pm}$,

(g) $T_{\phi,\mu}({}^zL^\nu_{\phi,\pm}) = {}^wL^\nu_{\mu,\pm}$.

Proof. The definition of the translation functor $T_{\phi,\mu} : \mathbf{O}_{\phi,\pm} \to \mathbf{O}_{\mu,\pm}$ in (3.3) is well-known, see e.g., [23], and, by construction, its restriction to Δ-filtered objects coincides with (3.2) if $A = k$. It satisfies the identities (e), (f) by [23, prop. 3.8]. Thus, since $z \in I_\mu^{\max}$, the functor $T_{\phi,\mu}$ factors to a functor as in (3.3), which satisfies again (e), (f). The existence of the left adjoint functor $T_{\mu,\phi}$ follows from the following general fact, see e.g., [37, lem. 2.8],

Claim 3.39. *Let A, B be noetherian k-algebras and $T : \mathbf{mod}(A) \to \mathbf{mod}(B)$ be a right exact k-linear functor which commutes with direct sums. Then T has a right adjoint.*

This implies that the functor $T_{\phi,\mu}$ in (3.3) has a right adjoint. Composing this right adjoint with the BGG duality we get a left adjoint, see Remark 2.15. Hence, claim (a) is proved. Part (b) follows from Proposition 3.37. Claim (c) is obvious, and (d) is a consequence of (e), (f). To prove (g), note that (e) implies that $T_{\phi,\mu}({}^zL_{\phi,\pm}) = {}^wL_{\mu,\pm}$. Thus (c) gives $T_{\phi,\mu}({}^zL^\nu_{\phi,\pm}) \subset {}^wL^\nu_{\mu,\pm}$. Further, for $x \in {}^zI^\nu_{\mu,\pm}$ part (e) yields

$$L(xw_\mu \bullet o_{\phi,\pm}) \subset T_{\mu,\phi}T_{\phi,\mu}L(xw_\mu \bullet o_{\phi,\pm}) = T_{\mu,\phi}L(x \bullet o_{\mu,\pm}). \tag{3.4}$$

Thus, by adjunction, for each $x \in {}^zI^\nu_{\mu,\pm}$ we have a surjective map

$$T_{\phi,\mu}L(xw_\mu \bullet o_{\phi,\pm}) \to L(x \bullet o_{\mu,\pm}).$$

Now, since the right hand side of (3.4) is in ${}^w\mathbf{O}^\nu_{\phi,\pm}$ by (c), the left hand side is also in ${}^z\mathbf{O}^\nu_{\phi,\pm}$. Thus, we have a surjective map $T_{\phi,\mu}({}^zL^\nu_{\phi,\pm}) \to {}^wL^\nu_{\mu,\pm}$. $\square$

3.5. **Translation functors in Z.** In this section we set $V=\mathbf{t}$ and we assume that A is the localization of S_0 with respect to some multiplicative subset. Fix $z\in I_\mu^{\max}$. An element of ${}^z\bar{Z}_{A,\phi}$ is a tuple (a_x) of elements of A with $x\leqslant z$. The assignment $y\cdot(a_x)=(a_{xy})$ defines a left W_μ-action on ${}^z\bar{Z}_{A,\phi}$. For $w\in zW_\mu$ the map

$$ {}^w\bar{Z}_{A,\mu}\to {}^z\bar{Z}_{A,\phi},\quad (a_x)\mapsto(a_{xy})\quad\text{with}\quad a_{xy}=a_x,\ x\in {}^wI_\mu^{\min},\ y\in W_\mu $$

identifies ${}^w\bar{Z}_{A,\mu}$ with the set of W_μ-invariant elements in ${}^z\bar{Z}_{A,\phi}$. Let

$$ \bar\theta_{\phi,\mu}:{}^z\bar{\mathbf{Z}}_{A,\phi}\to {}^w\bar{\mathbf{Z}}_{A,\mu},\quad \bar\theta_{\mu,\phi}:{}^w\bar{\mathbf{Z}}_{A,\mu}\to {}^z\bar{\mathbf{Z}}_{A,\phi} $$

be the restriction and induction functors with respect to the inclusion

$$ {}^w\bar{Z}_{A,\mu}\subset {}^z\bar{Z}_{A,\phi}. $$

Forgetting the gradings we define in the same way the functors $\theta_{\mu,\phi}$ and $\theta_{\phi,\mu}$.

Proposition 3.40. *For $z\in I_\mu^{\max}$ the following hold*
(a) $\bar\theta_{\phi,\mu}$, $\bar\theta_{\mu,\phi}$ commute with base change,
(b) $\mathbb{V}\circ T_{\phi,\mu}=\theta_{\phi,\mu}\circ\mathbb{V}$ and $\mathbb{V}\circ T_{\mu,\phi}=\theta_{\mu,\phi}\circ\mathbb{V}$,
(c) $\bar\theta_{\phi,\mu}$ and $\bar\theta_{\mu,\phi}$ are exact functors ${}^z\bar{\mathbf{Z}}^\Delta_{A,\phi,\pm}\to {}^w\bar{\mathbf{Z}}^\Delta_{A,\mu,\pm}$ and ${}^w\bar{\mathbf{Z}}^\Delta_{A,\mu,\pm}\to {}^z\bar{\mathbf{Z}}^\Delta_{A,\phi,\pm}$,
(d) $(\bar\theta_{\mu,\phi},\bar\theta_{\phi,\mu},\bar\theta_{\mu,\phi}\{2l(w_\mu)\})$ is a triple of adjoint functors,
(e) $\bar\theta_{\phi,\mu}\circ D=D\circ\bar\theta_{\phi,\mu}$ and $\bar\theta_{\mu,\phi}\circ D=D\circ\bar\theta_{\mu,\phi}\circ\{2l(w_\mu)\}$,
(f) for $x\in {}^wI_\mu^{\min}$, there is a sum M of ${}^w\bar{B}_{A,\mu,+}(t)\{j\}$'s with $t<x$ such that

$$ \bar\theta_{\phi,\mu}({}^z\bar{B}_{A,\phi,+}(xw_\mu))=\bigoplus_{y\in W_\mu}{}^w\bar{B}_{A,\mu,+}(x)\{l(w_\mu)-2l(y)\}\oplus M, $$

(g) for $x\in {}^wI_\mu^{\min}$ we have

$$ \begin{aligned} \bar\theta_{\mu,\phi}({}^w\bar{B}_{A,\mu,-}(x))&={}^z\bar{B}_{A,\phi,-}(x),\\ \bar\theta_{\mu,\phi}({}^w\bar{B}_{A,\mu,+}(x))&={}^z\bar{B}_{A,\phi,+}(xw_\mu)\{-l(w_\mu)\}. \end{aligned} $$

Proof. Part (a) is obvious. Part (b) is proved in [14, thm. 9]. For (c) it is enough to check that $\theta_{\phi,\mu}$, $\theta_{\mu,\phi}$ preserve ${}^z\mathbf{Z}^\Delta_{A,\phi,\pm}$, ${}^w\mathbf{Z}^\Delta_{A,\mu,\pm}$ and are exact. This follows from claim (b) and Propositions 3.32(c), 3.37(a). Part (d) is proved as in [16, prop. 5.2]. More precisely, since $z\in I_\mu^{\max}$ there are graded ${}^w\bar{Z}_{A,\mu}$-module isomorphisms

$$ {}^z\bar{Z}_{A,\phi}\ \simeq\ \bigoplus_{y\in W_\mu}{}^w\bar{Z}_{A,\mu}\{-2l(y)\}, \tag{3.5} $$

$$ {}^z\bar{Z}_{A,\phi}\{2l(w_\mu)\}\ \simeq\ \mathrm{Hom}_{{}^w\bar{Z}_{A,\mu}}\big({}^z\bar{Z}_{A,\phi},{}^w\bar{Z}_{A,\mu}\big). \tag{3.6} $$

The second one yields an isomorphism of functors

$$ \begin{aligned} {}^z\bar{Z}_{A,\phi}\{2l(w_\mu)\}\otimes_{{}^w\bar{Z}_{A,\mu}}\bullet\ &\simeq\ \mathrm{Hom}_{{}^w\bar{Z}_{A,\mu}}\big({}^z\bar{Z}_{A,\phi},{}^w\bar{Z}_{A,\mu}\big)\otimes_{{}^w\bar{Z}_{A,\mu}}\bullet\\ &\simeq\ \mathrm{Hom}_{{}^w\bar{Z}_{A,\mu}}\big({}^z\bar{Z}_{A,\phi},\bullet\big). \end{aligned} $$

Therefore $(\bar\theta_{\mu,\phi},\bar\theta_{\phi,\mu},\bar\theta_{\mu,\phi}\{2l(w_\mu)\})$ is a triple of adjoint of functors. Part (e) follows from (d). Indeed, since $\bar\theta_{\phi,\mu}(M)=M$ as a graded A-module, we have $\bar\theta_{\phi,\mu}\circ D=D\circ\bar\theta_{\phi,\mu}$. Then, part (d) implies that $\bar\theta_{\mu,\phi}\circ D=D\circ\bar\theta_{\mu,\phi}\circ\{2l(w_\mu)\}$. Now, we prove (f). We abbreviate $\bar{B}_\mu(x)={}^w\bar{B}_{A,\mu,+}(x)$, $\bar{B}_\phi(x)={}^z\bar{B}_{A,\phi,+}(x)$, $\bar{Z}_\mu={}^w\bar{Z}_{A,\mu}$ and $\bar{Z}_\phi={}^z\bar{Z}_{A,\phi}$. We have $\bar{B}_\phi(xw_\mu)_{xW_\mu}=\bar{Z}_{\phi,xW_\mu}\{l(xw_\mu)\}$, see Remark 3.45. Thus, by [16, prop. 5.3] and (3.5) we have also

$$ \bar\theta_{\phi,\mu}(\bar{B}_\phi(xw_\mu))_x=\bar{B}_\phi(xw_\mu)_{xW_\mu}=\bigoplus_{y\in W_\mu}A\{l(xw_\mu)-2l(y)\} $$

as a graded A-module. This proves the claim, because for $t\in I_\mu^{\min}$ we have

$$ t\not\leqslant x\Rightarrow\bar\theta_{\phi,\mu}(\bar{B}_\phi(xw_\mu))_t=\bar\theta_{\phi,\mu}(\bar{B}_\phi(xw_\mu)_{tW_\mu})=0, $$

and because $\bar\theta_{\phi,\mu}(\bar B_\phi(xw_\mu))$ is a direct sum of objects of the form $\bar B_\mu(z)\{j\}$ since the right adjoint of $\bar\theta_{\phi,\mu}$ is exact. Finally, we prove (g). By Propositions 3.32(c), 3.38(d) and part (b) we have

$$\theta_{\mu,\phi}({}^wB_{A,\mu,+}(x)) = {}^zB_{A,\phi,+}(xw_\mu), \quad \theta_{\mu,\phi}({}^wB_{A,\mu,-}(x)) = {}^zB_{A,\phi,-}(x).$$

To identify the gradings, note that by [16, prop. 5.3], we have

$$\begin{aligned}\bar\theta_{\mu,\phi}({}^w\bar B_{A,\mu,\pm}(x))_{xW_\mu} &= {}^z\bar Z_{\phi,xW_\mu} \otimes_{{}^w\bar Z_{\mu,x}} {}^w\bar B_{A,\mu,\pm}(x)_x \\ &= {}^z\bar Z_{\phi,xW_\mu}\{\pm l(x)\},\end{aligned}$$

because ${}^w\bar Z_{\mu,x} = A$ and ${}^w\bar B_{A,\mu,\pm}(x)_x = A\{\pm l(x)\}$. Therefore, we have

$$\bar\theta_{\mu,\phi}({}^w\bar B_{A,\mu,+}(x))_{xw_\mu} = A\{l(x)\} \quad \bar\theta_{\mu,\phi}({}^w\bar B_{A,\mu,-}(x))_x = A\{-l(x)\},$$

because $({}^z\bar Z_{\phi,xW_\mu})_y = A$ for all $y \in xW_\mu$. □

Remark 3.41. The isomorphisms (3.5), (3.6) are only proved for $\sharp W_\mu = 2$ in [16, lem. 5.1]. The general case is similar and is left to the reader.

Remark 3.42. For $z \in I_\mu^{\max}$, $w \in zW_\mu$ and $e + m > d$ the functor $T_{\mu,\phi} : {}^w\mathbf{O}_{\mu,+} \to {}^z\mathbf{O}_{\phi,+}$ is faithful on projectives. By Propositions 3.40(b), 3.32(d) it is enough to check that $\theta_{\mu,\phi}$ is faithful on the projective objects in ${}^w\mathbf{Z}_{\mu,+}$. This is obvious, because the unit $\mathbf{1} \to \theta_{\mu,\phi} \circ \theta_{\phi,\mu}$ is a direct summand by definition of $\theta_{\mu,\phi}$, $\theta_{\phi,\mu}$.

Remark 3.43. If $A = k$ then the functors $\bar\theta_{\mu,\phi}$, $\bar\theta_{\phi,\mu}$ and their non graded analogues are defined in the following way. Recall that ${}^w\bar Z_\mu = k{}^w\bar Z_{S_0,\mu}$ and that ${}^w\bar{\mathbf{Z}}_\mu$ is the category of the finite dimensional graded ${}^w\bar Z_\mu$-modules. Then, we define

$$\bar\theta_{\phi,\mu} : {}^z\bar{\mathbf{Z}}_\phi \to {}^w\bar{\mathbf{Z}}_\mu, \quad \bar\theta_{\mu,\phi} : {}^w\bar{\mathbf{Z}}_\mu \to {}^z\bar{\mathbf{Z}}_\phi$$

to be the restriction and induction functors with respect to the inclusion ${}^w\bar Z_\mu \subset {}^z\bar Z_\phi$. These functors $\bar\theta_{\mu,\phi}$, $\bar\theta_{\phi,\mu}$ are exact, for the obvious exact structure on $\mathbf{gmod}({}^w\bar Z_\mu)$.

3.6. **Localization.** In this section we set $V = \mathbf{t}^*$. Let P_μ be the "parabolic subgroup" of $G(k((t)))$ with Lie algebra $\mathbf{p}_\mu$. Write $B = P_\phi$. Let $T \subset B$ be the torus associated with $\mathbf{t}$.

Let $X' = G(k((t)))/P_\mu$ be the partial (affine) flag ind-scheme. For $w \in \widehat W$ let $\bar X_w \subset X'$ be the corresponding finite dimensional affine Schubert variety. To avoid confusions we may write $X'_\mu = X'$ and $\bar X_{\mu,w} = \bar X_w$. The group T acts on $\bar X_w$, with the first copy of $k^\times$ acting by rotating the loop and the last one acting trivially. The varieties $\bar X_w$ form an inductive system of complex projective varieties with closed embeddings and X' is represented by the ind-scheme $\mathrm{ind}_w \bar X_w$. Let $\mathbf{D}^b(\bar X_w)$ be the bounded derived category of constructible sheaves of k-vector spaces on $\bar X_w$ which are locally constant along the B-orbits. Let $\mathbf{P}(\bar X_w)$ be the full subcategory of perverse sheaves. Recall that $\bar X_w$ has dimension $l(w)$ for $w \in I_\mu^{\min}$.

For $x \in I_{\mu,+}$ we have $\bar X_x \subset \bar X_w$ if and only if $x \in {}^wI_{\mu,+}$. Let ${}^wIC(\bar X_x)$ be the intersection cohomology complex in $\mathbf{P}(\bar X_w)$ associated with $\bar X_x$. Let $IH(\bar X_x)$ (resp. $IH_T(\bar X_x)$) be the (resp. T-equivariant) intersection cohomology of $\bar X_x$. See Section A.1 for details.

Proposition 3.44. *We have*
(a) $H_T(\bar X_w) = {}^w\bar Z^\vee_{S,\mu}$ *and* $H(\bar X_w) = {}^w\bar Z^\vee_\mu$ *as graded k-algebras,*
(b) $IH_T(\bar X_x) = {}^w\bar B^\vee_{S,\mu,+}(x)$ *as a graded* ${}^w\bar Z^\vee_{S,\mu}$*-module,*
(c) $IH(\bar X_x) = {}^w\bar B^\vee_{\mu,+}(x)$ *as a graded* ${}^w\bar Z^\vee_\mu$*-module.*

Proof. Part (a) follows from [20], parts (b) and (c) from [7, thm. 1.5, 1.6, 1.8]. □

Remark 3.45. By Proposition 3.44, the graded ${}^z\bar{Z}_{\phi,xW_\mu}$-module ${}^z\bar{B}_{A,\phi,+}(xw_\mu)_{xW_\mu}$ with $x \in I_\mu^{\min}$ is the equivariant intersection cohomology of the variety $\bigcup_{y\in W_\mu} X_{\phi,xy}$. Since this variety is smooth, we have ${}^z\bar{B}_{A,\phi,+}(xw_\mu)_{xW_\mu} = {}^z\bar{Z}_{\phi,xW_\mu}\{l(xw_\mu)\}$.

Proposition 3.46. *Let $w \in I_{\mu,+}$ and $v = w_-^{-1}$, so $v \in I^\mu_{\phi,-}$. There is an equivalence of abelian categories $\Phi : {}^v\mathbf{O}^\mu_{\phi,-} \to \mathbf{P}(\bar{X}_w)$ such that ${}^vL(y \bullet \mathrm{o}_{\phi,-}) \mapsto {}^wIC(\bar{X}_x)$ for $y \in {}^vI^\mu_{\phi,-}$ and $x = y_+^{-1}$.*

Proof. Note that $w = v^{-1}w_\mu$ and that $x = y^{-1}w_\mu$. Further, the assignment $y \mapsto x$ yields a bijection ${}^vI^\mu_{\phi,-} \to {}^wI_{\mu,+}$ by Remark 2.12. Next, apply [4, thm. 7.15, 7.16], [18, thm. 2.2] and [26]. □

Recall the category ${}^w\mathbf{Z}^\vee_\mu$ from Remark 3.28. By Proposition 3.44, composing Φ and the cohomology, for $w \in I_{\mu,+}$ and $v = w_-^{-1} = w_\mu w^{-1} \in I^\mu_{\phi,-}$, we get a functor

$$\mathbb{H} : {}^v\mathbf{O}^\mu_{\phi,-} \to {}^w\mathbf{Z}^\vee_\mu.$$

Proposition 3.47. *Let $w \in I_{\mu,+}$ and $y, t \in {}^vI^\mu_{\phi,-}$. Set $v = w_\mu w^{-1}$, $x = y_+^{-1}$ and $s = t_+^{-1}$. We have*

(a) $\mathbb{H}({}^vL(y \bullet \mathrm{o}_{\phi,-})) = {}^wB^\vee_{\mu,+}(x)$,

(b) $\mathrm{Ext}_{{}^v\mathbf{O}^\mu_{\phi,-}}\big({}^vL(y \bullet \mathrm{o}_{\phi,-}), {}^vL(t \bullet \mathrm{o}_{\phi,-})\big) = k\,\mathrm{Hom}_{{}^wZ^\vee_{S,\mu}}\big({}^w\bar{B}^\vee_{S,\mu,+}(x), {}^w\bar{B}^\vee_{S,\mu,+}(s)\big)$,

(c) For $z = v^{-1}$ and $L \in \mathrm{Irr}({}^v\mathbf{O}^\mu_{\phi,-})$ we have an isomorphism of ${}^zZ^\vee_\phi$-modules

$$\theta_{\mu,\phi}\mathbb{H}(L) \simeq \mathbb{H}i(L).$$

Proof. Part (a) follows from Propositions 3.44, 3.46. Next, Proposition 3.46 gives

$$\mathrm{Ext}_{{}^v\mathbf{O}^\mu_{\phi,-}}\big({}^vL(y \bullet \mathrm{o}_{\phi,-}), {}^vL(t \bullet \mathrm{o}_{\phi,-})\big) = \mathrm{Ext}_{\mathbf{D}^b(\bar{X}_w)}\big({}^wIC(\bar{X}_x), {}^wIC(\bar{X}_s)\big).$$

Further, by Propositions A.1 and 3.44 we have

$$\mathrm{Ext}_{\mathbf{D}^b_T(\bar{X}_w)}\big({}^wIC_T(\bar{X}_x), {}^wIC_T(\bar{X}_s)\big) = \mathrm{Hom}_{{}^wZ^\vee_{S,\mu}}\big({}^w\bar{B}^\vee_{S,\mu,+}(x), {}^w\bar{B}^\vee_{S,\mu,+}(s)\big),$$

$$\mathrm{Ext}_{\mathbf{D}^b(\bar{X}_w)}\big({}^wIC(\bar{X}_x), {}^wIC(\bar{X}_s)\big) = k\,\mathrm{Ext}_{\mathbf{D}^b_T(\bar{X}_w)}\big({}^wIC_T(\bar{X}_x), k\,{}^wIC_T(\bar{X}_s)\big).$$

This proves (b). Finally, we prove (c). Note that $\theta_{\mu,\phi}$ is well-defined, because $z \in I_\mu^{\max}$ and $w = zw_\mu$. By Proposition 3.44, taking the cohomology gives a functor $\mathbf{P}(\bar{X}_{\mu,w}) \to \mathbf{mod}({}^w\bar{Z}^\vee_\mu)$, $\mathcal{E} \mapsto H(\mathcal{E})$. Since $z \in I_\mu^{\max}$, the obvious projection $p : \bar{X}_{\phi,z} \to \bar{X}_{\mu,w}$ is a smooth map. For $\mathcal{E}$ in $\mathbf{P}(\bar{X}_{\mu,w})$ we may regard $H(\mathcal{E})$ and $H(p^*\mathcal{E})$ as modules over ${}^w\bar{Z}^\vee_\mu$ and ${}^z\bar{Z}^\vee_\phi$ by Proposition 3.44. There is a natural morphism $H(\mathcal{E}) \to \theta_{\phi,\mu}H(p^*\mathcal{E})$. By adjunction, this yields a morphism of functors $\theta_{\mu,\phi}\circ\mathbb{H} \to \mathbb{H}\circ i$. This is an isomorphism, by Proposition 3.40(g) via base change. □

Corollary 3.48. *Let $w \in I_{\mu,+}$ and $v = w_-^{-1}$. We have a graded k-algebra isomorphism ${}^v\bar{R}^\mu_{\phi,-} \to {}^w\bar{A}_{\mu,+}$ such that $1_y \mapsto 1_x$ with $x = y_+^{-1}$.*

Proof. The choice of a $\widehat{W}$-invariant pairing on $\mathbf{t}$ yields an isomorphism

$$k\,\mathrm{End}_{{}^wZ^\vee_{S,\mu}}({}^w\bar{B}^\vee_{S,\mu,+})^{\mathrm{op}} = {}^w\bar{A}_{\mu,+}.$$

Thus the corollary follows from Proposition 3.47. □

4. Proof of the main theorem

4.1. **Regular case.** Fix $d, e > 0$ and $\mu \in \mathcal{P}$. Let $\mathrm{o}_{\mu,-}$, $\mathrm{o}_{\phi,-}$ have the level $-e-m$ and $-d-m$. Let $w \in I_{\mu,+}$ and $z = ww_\mu$, $v = w_\mu w^{-1}$, $u = w^{-1}$. Note that

$$z \in I_{\mu,-}, \quad v \in I^\mu_{\phi,-}, \quad u \in I^\mu_{\phi,+}.$$

Proposition 4.1. *We have a k-algebra isomorphism ${}^wR_{\mu,+} = {}^v\bar{R}^\mu_{\phi,-}$ such that $1_x \mapsto 1_y$ for $x \in {}^wI_{\mu,+}$ and $y = x_-^{-1}$ in ${}^vI^\mu_{\phi,-}$. We have a k-algebra isomorphism ${}^w\bar{R}_{\mu,+} = {}^vR^\mu_{\phi,-}$ such that $1_x \mapsto 1_y$ for $x \in {}^wI_{\mu,+}$. The graded k-algebras ${}^w\bar{R}_{\mu,+}$ and ${}^v\bar{R}^\mu_{\phi,-}$, are Koszul. Further ${}^w\bar{R}_{\mu,+} = E({}^v\bar{R}^\mu_{\phi,-})$ and $1_x = E(1_y)$ for $x \in {}^wI_{\mu,+}$.*

Proof. Corollaries 3.33, 3.48 yield k-algebra isomorphisms

$${}^wR_{\mu,+} = {}^wA_{\mu,+} = {}^v\bar{R}^\mu_{\phi,-}$$

which identify $1_y \in {}^v\bar{R}^\mu_{\phi,-}$ with $1_x \in {}^wR_{\mu,+}$.

Now, we claim that the k-algebra ${}^vR^\mu_{\phi,-}$ has a Koszul grading. By Lemma 2.2 and Remark 2.13, it is enough to check that ${}^vR_{\phi,-}$ has a Koszul grading. This follows from the matrix equation in Proposition 3.25 and from [5, thm. 2.11.1], because ${}^vR_{\phi,-} = {}^vA_{\phi,-}$ as k-algebras by Corollary 3.33.

Equip ${}^vR^\mu_{\phi,-}$ with the Koszul grading above. Then Lemma 2.1 implies that $E({}^vR^\mu_{\phi,-}) = {}^v\bar{R}^\mu_{\phi,-}$. Thus ${}^v\bar{R}^\mu_{\phi,-}$ is Koszul. Since ${}^wR_{\mu,+} = {}^v\bar{R}^\mu_{\phi,-}$, this implies that ${}^wR_{\mu,+}$ has a Koszul grading. Thus Lemma 2.1 gives $E({}^wR_{\mu,+}) = {}^w\bar{R}_{\mu,+}$. Hence ${}^w\bar{R}_{\mu,+}$ is Koszul. Finally, we have k-algebra isomorphisms

$${}^vR^\mu_{\phi,-} = E({}^v\bar{R}^\mu_{\phi,-}) = E({}^wR_{\mu,+}) = {}^w\bar{R}_{\mu,+}.$$

They identify the idempotent $1_y \in {}^vR^\mu_{\phi,-}$ with the idempotent $1_x \in {}^w\bar{R}_{\mu,+}$. $\square$

Remark 4.2. The Koszul grading on ${}^vR^\mu_{\phi,-}$ can also be obtained using mixed perverse sheaves on the ind-scheme X' as in [5, thm. 4.5.4], [1]. Our argument via moment graphs is elementary. Note that there is no analogue, in our situation, of [5, lem. 3.9.2], because ${}^vR_{\phi,-}$ is not Koszul self-dual. Note also that there is no analogue of the localization functor Φ in Proposition 3.46 for positive levels.

For the next proposition we use standard Koszul duality technics. To do so, we need the following result.

Lemma 4.3. *The quasi-hereditary k-algebra ${}^wR_{\mu,+}$ is balanced.*

Now we can prove the second main result of this section.

Proposition 4.4. *We have a k-algebra isomorphism ${}^z\bar{R}_{\mu,-} = {}^uR^\mu_{\phi,+}$ such that $1_x \mapsto 1_y$ for $x \in {}^zI_{\mu,-}$ and $y = x_+^{-1}$ in ${}^uI^\mu_{\phi,+}$. We have a k-algebra isomorphism ${}^zR_{\mu,-} = {}^u\bar{R}^\mu_{\phi,+}$ such that $1_x \mapsto 1_y$ for $x \in {}^zI_{\mu,-}$. The graded k-algebras ${}^u\bar{R}^\mu_{\phi,+}$ and ${}^z\bar{R}_{\mu,-}$ are Koszul. Further ${}^z\bar{R}_{\mu,-} = E({}^u\bar{R}^\mu_{\phi,+})$ and $1_x = E(1_y)$ for $x \in {}^zI_{\mu,-}$.*

Proof. By Proposition 2.10 we have $D({}^wR_{\mu,+}) = {}^zR_{\mu,-}$. Thus ${}^zR_{\mu,-}$ has a Koszul grading by Lemma 4.3. Next, Lemma 2.1 implies that ${}^z\bar{R}_{\mu,-} = E({}^zR_{\mu,-})$ is Koszul. Thus [30, thm. 1] and Propositions 2.10, 4.1 yield a k-algebra isomorphism

$${}^z\bar{R}_{\mu,-} = E({}^zR_{\mu,-}) = DED({}^zR_{\mu,-}) = DE({}^wR_{\mu,+}) = D({}^w\bar{R}_{\mu,+}) = D({}^vR^\mu_{\phi,-}) = {}^uR^\mu_{\phi,+}$$

such that $1_x \mapsto 1_y$. Next, by [30, thm. 1] and the isomorphisms above, ${}^uR^\mu_{\phi,+}$ has a Koszul grading and is balanced. Hence $E({}^uR^\mu_{\phi,+}) = {}^u\bar{R}^\mu_{\phi,+}$ is Koszul by Lemma 2.1. So we have

$${}^u\bar{R}^\mu_{\phi,+} = E({}^uR^\mu_{\phi,+}) = DED({}^uR^\mu_{\phi,+}) = DE({}^vR^\mu_{\phi,-}) = D({}^v\bar{R}^\mu_{\phi,-}) = D({}^wR_{\mu,+}) = {}^zR_{\mu,-}.$$

$\square$

Proof of Lemma 4.3. We equip the quasi-hereditary k-algebra ${}^wR_{\mu,+}$ with the Koszul grading ${}^v\bar{R}^\mu_{\phi,-}$, see Proposition 4.1. The Koszul dual $E({}^v\bar{R}^\mu_{\phi,-}) = {}^w\bar{R}_{\mu,+}$ is quasi-hereditary, because it is isomorphic to ${}^vR^\mu_{\phi,-}$ as a k-algebra. Since it is also Koszul, this implies that the graded k-algebra ${}^v\bar{R}^\mu_{\phi,-}$ is standard Koszul, see Section 2.6. Therefore, by [29, thm. 6], we must prove that the grading on $D({}^v\bar{R}^\mu_{\phi,-})$ is positive. As k-algebras we have

$$D({}^v\bar{R}^\mu_{\phi,-}) = D({}^wR_{\mu,+}) = \mathrm{End}_{{}^wR_{\mu,+}}({}^wT_{\mu,+})^{\mathrm{op}}.$$

As graded k-algebras we have

$$D({}^v\bar{R}^\mu_{\phi,-}) = \mathrm{End}_{{}^wR_{\mu,+}}({}^w\bar{T}_{\mu,+})^{\mathrm{op}}.$$

Here ${}^w\bar{T}_{\mu,+}$ is the ${}^v\bar{R}^\mu_{\phi,-}$-module equal to ${}^wT_{\mu,+}$ as a ${}^wR_{\mu,+}$-module, with the natural grading. By Corollary 3.48 we have a graded k-algebra isomorphism

$${}^v\bar{R}^\mu_{\phi,-} = {}^w\bar{A}_{\mu,+}.$$

We claim that there is also a graded k-algebra isomorphism

$$D({}^v\bar{R}^\mu_{\phi,-}) = {}^z\bar{A}_{\mu,-}. \tag{4.1}$$

This implies the lemma because ${}^z\bar{A}_{\mu,-}$ is positively graded by Proposition 3.25. To prove the claim, observe first that Propositions 2.10, 3.31, 3.32(c) yield a k-algebra isomorphism

$$\begin{aligned}
D({}^wR_{\mu,+}) &= {}^zR_{\mu,-}\\
&= \mathrm{End}_{{}^z\mathbf{O}_{\mu,-}}({}^zP_{\mu,-})^{\mathrm{op}}\\
&= k\,\mathrm{End}_{{}^z\mathbf{O}_{S_0,\mu,-}}({}^zP_{S_0,\mu,-})^{\mathrm{op}}\\
&= k\,\mathrm{End}_{{}^z\mathbf{O}_{S_0,\mu,+}}({}^zT_{S_0,\mu,+})\\
&= k\,\mathrm{End}_{{}^zZ_{S_0,\mu}}({}^zC_{S_0,\mu,+})\\
&= {}^zA_{\mu,-}.
\end{aligned}$$

So we just have to identify the gradings in (4.1). Set

$$\begin{aligned}
{}^w\bar{A}_{S_0,\mu,+} &= \mathrm{End}_{{}^wZ_{S_0,\mu}}\big({}^w\bar{B}_{S_0,\mu,+}\big)^{\mathrm{op}},\\
{}^wA_{S_0,\mu,+} &= \mathrm{End}_{{}^wZ_{S_0,\mu}}\big({}^wB_{S_0,\mu,+}\big)^{\mathrm{op}}.
\end{aligned}$$

By Proposition 3.32(c) we have

$${}^w\mathbf{O}_{S_0,\mu,+} = \mathbf{mod}({}^wA_{S_0,\mu,+}).$$

Consider the graded S_0-category

$${}^w\bar{\mathbf{O}}_{S_0,\mu,+} = \mathbf{gmod}({}^w\bar{A}_{S_0,\mu,+}).$$

Let ${}^w\bar{\mathbf{O}}^\Delta_{S_0,\mu,+}$ be the full subcategory of ${}^w\bar{\mathbf{O}}_{S_0,\mu,+}$ of the modules taken to ${}^w\mathbf{O}^\Delta_{S_0,\mu,+}$ by the canonical functor ${}^w\bar{\mathbf{O}}_{S_0,\mu,+} \to {}^w\mathbf{O}_{S_0,\mu,+}$. Consider the following square

$$\begin{tikzcd}
{}^w\bar{\mathbf{O}}^\Delta_{S_0,\mu,+} \arrow[r, "\bar{\mathbb{V}}"'] \arrow[d] & {}^w\bar{\mathbf{Z}}^\Delta_{S_0,\mu,+} \arrow[d]\\
{}^w\mathbf{O}^\Delta_{S_0,\mu,+} \arrow[r, "\mathbb{V}"] & {}^w\mathbf{Z}^\Delta_{S_0,\mu,+}.
\end{tikzcd} \tag{4.2}$$

The vertical arrows are the obvious ones and $\bar{\mathbb{V}}$ is the functor given by

$$\bar{\mathbb{V}}(M) = {}^w\bar{B}_{S_0,\mu,+} \otimes_{{}^w\bar{A}_{S_0,\mu,+}} M.$$

The square (4.2) is commutative. Indeed, view ${}^w\mathbf{O}^\Delta_{S_0,\mu,+}$ as a subcategory of

$${}^w\mathbf{O}_{S_0,\mu,+} = \mathbf{mod}({}^wA_{S_0,\mu,+}).$$

Under this identification the module ${}^wP_{S_0,\mu,+}$ is taken to ${}^wA_{S_0,\mu,+}$. Now, $\mathbb{V}$ is a right exact functor which takes ${}^wA_{S_0,\mu,+}$ to ${}^wB_{S_0,\mu,+}$. Thus, under the identification above, for each M in ${}^w\mathbf{O}^{\Delta}_{S_0,\mu,+}$ we have

$$\mathbb{V}(M) = {}^wB_{S_0,\mu,+} \otimes_{{}^wA_{S_0,\mu,+}} M.$$

This means precisely that the square (4.2) is commutative. Note that $\bar{\mathbb{V}}$ is fully faithful on ${}^w\bar{\mathbf{O}}^{\Delta}_{S_0,\mu,+}$ because $\mathbb{V}$ is fully faithful on ${}^w\mathbf{O}^{\Delta}_{S_0,\mu,+}$ by Proposition 3.32(c). Next, by Corollary A.6 we have a graded k-algebra isomorphism

$${}^z\bar{A}_{\mu,-} = \mathrm{End}_{{}^wZ_\mu}\big({}^w\bar{C}_{\mu,+}\big).$$

Hence we must prove that there is a graded k-algebra isomorphism

$$\mathrm{End}_{{}^wR_{\mu,+}}({}^w\bar{T}_{\mu,+}) = \mathrm{End}_{{}^wZ_\mu}\big({}^w\bar{C}_{\mu,+}\big).$$

By Remark 3.35 it is enough to check that $\bar{\mathbb{V}}({}^w\bar{T}_{\mu,+}) = {}^w\bar{C}_{\mu,+}$. To do so, we consider the deformed tilting module ${}^wT_{S_0,\mu,+}$ in ${}^w\mathbf{O}_{S_0,\mu,+}$. It admits a natural grading. Equipped with this grading ${}^wT_{S_0,\mu,+}$ can be regarded as a graded ${}^w\bar{A}_{S_0,\mu,+}$-module ${}^w\bar{T}_{S_0,\mu,+}$, or, equivalently, an object in ${}^w\bar{\mathbf{O}}_{S_0,\mu,+}$. Since the construction of the natural grading commutes with the reduction to the field k, we have

$${}^w\bar{T}_{\mu,+} = k\,{}^w\bar{T}_{S_0,\mu,+}$$

and since $\bar{\mathbb{V}}$ commutes with base change, it is enough to check that

$$\bar{\mathbb{V}}({}^w\bar{T}_{S_0,\mu,+}) = {}^w\bar{C}_{S_0,\mu,+}. \tag{4.3}$$

By Proposition 3.32(c) we just have to identify the gradings in (4.3).

Now, recall that ${}^wT_{S_0}(x \bullet \mathrm{o}_{\mu,+})$ is filtered by deformed Verma modules and that the subquotients of this filtration are either $V_{S_0}(x\bullet\mathrm{o}_{\mu,+})$ or of the form $V_{S_0}(y\bullet\mathrm{o}_{\mu,+})$ with $y > x$. Let $\bar{V}_{S_0}(x\bullet\mathrm{o}_{\mu,+})$ be the natural grading on $V_{S_0}(x\bullet\mathrm{o}_{\mu,+})$. The inclusion

$$\bar{V}_{S_0}(x \bullet \mathrm{o}_{\mu,+}) \subset {}^w\bar{T}_{S_0}(x \bullet \mathrm{o}_{\mu,+})$$

is homogeneous of degree 0 by definition of the natural gradings. Next, by Remark 3.19 the graded S_0-sheaf ${}^w\bar{C}_{S_0,\mu,+}(x)$ is filtered by Verma-sheaves, and this filtration yields an inclusion

$$\bar{V}_{S_0}(x)\{l(x)\} \subset {}^w\bar{C}_{S_0,\mu,+}(x).$$

Therefore, since the grading of an indecomposable object is unique up to a grading shift, to prove (4.3) it is enough to check that

$$\bar{\mathbb{V}}\big(\bar{V}_{S_0}(x \bullet \mathrm{o}_{\mu,+})\big) = \bar{V}_{S_0}(x)\{l(x)\}.$$

Since $\mathbb{V}\big(V_{S_0}(x \bullet \mathrm{o}_{\mu,+})\big) = V_{S_0}(x)$ by Proposition 3.32(c), we just have to check the gradings, once again. But, by definition, we have

$${}^w\bar{P}_{S_0}(x \bullet \mathrm{o}_{\mu,+}) = {}^w\bar{A}_{S_0,\mu,+}1_x,$$

for some idempotent 1_x which is homogeneous of degree 0. Thus, by definition of $\bar{\mathbb{V}}$, we have

$$\bar{\mathbb{V}}\big({}^w\bar{P}_{S_0}(x \bullet \mathrm{o}_{\mu,+})\big) = {}^w\bar{B}_{S_0,\mu,+}(x).$$

Further, by Remark 3.19 the graded S_0-sheaf ${}^w\bar{B}_{S_0,\mu,+}(x)$ is filtered by Verma-sheaves, and this filtration yields a surjection

$${}^w\bar{B}_{S_0,\mu,+}(x) \to \bar{V}_{S_0}(x)\{l(x)\}.$$

Since the surjection

$${}^w\bar{P}_{S_0}(x \bullet \mathrm{o}_{\mu,+}) \to \bar{V}_{S_0}(x \bullet \mathrm{o}_{\mu,+})$$

is also homogeneous of degree 0 by definition of the natural gradings, we get our claim, proving the lemma. □

Corollary 4.5. *We have an isomorphism of graded k-algebras* ${}^u\bar{R}^{\mu}_{\phi,+} = {}^z\bar{A}_{\mu,-}$.

Proof. By [30, thm. 1] and the proof of Lemma 4.3 the graded k-algebra

$$D({}^v\bar{R}^\mu_{\phi,-}) = {}^z\bar{A}_{\mu,-}$$

is Koszul. By Proposition 4.4 we have that ${}^u\bar{R}^\mu_{\phi,+}$ is Koszul and is isomorphic to ${}^wR_{\mu,-}$ as a k-algebra. Further ${}^v\bar{R}^\mu_{\phi,-} = {}^wR_{\mu,+}$ as a k-algebra and $D({}^wR_{\mu,+}) = {}^zR_{\mu,-}$. Thus we have a k-algebra isomorphism

$${}^zR_{\mu,-} = D({}^wR_{\mu,+}) = {}^zA_{\mu,-},$$

which lifts to a graded k-algebra isomorphism ${}^u\bar{R}^\mu_{\phi,+} = {}^z\bar{A}_{\mu,-}$ by unicity of the Koszul grading. □

4.2. **General case.** We can now complete the proof of Theorem 2.16. We first prove a series of preliminary lemmas. Fix $d, e, f > 0$, $\mu, \nu \in \mathcal{P}$ and $w \in \widehat{W}$. Choose integral weights $o_{\mu,-}$, $o_{\nu,-}$, $o_{\phi,-}$ of level $-e-m$, $-f-m$ and $-d-m$.

Lemma 4.6. *For $w \in {}^wI^\nu_{\mu,+}$ the functor τ yields a surjective k-algebra homomorphism $\tau_{\phi,\nu} : {}^wR_{\mu,+} \to {}^wR^\nu_{\mu,+}$. The kernel of $\tau_{\phi,\nu}$ is the two-sided ideal generated by the idempotents 1_x with $x \in {}^wI_{\mu,+} \setminus {}^wI^\nu_{\mu,+}$, and we have $\tau_{\phi,\nu}(1_x) = 1_x$ for $x \in {}^wI^\nu_{\mu,+}$.*

Proof. By Remark 2.13 the functor τ takes the full projective module of ${}^w\mathbf{O}_{\mu,+}$ to the full projective module of ${}^w\mathbf{O}^\nu_{\mu,+}$. For any M the unit $M \to i\tau(M)$ is surjective. Hence for P projective we have a surjective map

$$\mathrm{Hom}_{{}^w\mathbf{O}_{\mu,+}}(P, M) \to \mathrm{Hom}_{{}^w\mathbf{O}_{\mu,+}}(P, i\tau(M)) = \mathrm{Hom}_{{}^w\mathbf{O}^\nu_{\mu,+}}(\tau(P), \tau(M)).$$

Thus τ yields a surjective k-algebra homomorphism $\tau_{\phi,\nu} : {}^wR_{\mu,+} \to {}^wR^\nu_{\mu,+}$. Let $I \subset {}^wR_{\mu,+}$ be the 2-sided ideal generated by the idempotents 1_x, $x \in {}^wI_{\mu,+}$, such that $\tau_{\phi,\nu}(1_x) = 0$. By Remark 2.13(c) the latter are precisely the idempotents 1_x with $x \in {}^wI_{\mu,+} \setminus {}^wI^\nu_{\mu,+}$. We have a k-algebra isomorphism ${}^wR_{\mu,+}/I \simeq {}^wR^\nu_{\mu,+}$, because ${}^w\mathbf{O}^\nu_{\mu,+}$ is the thick subcategory of ${}^w\mathbf{O}_{\mu,+}$ generated by the simple modules killed by I. It is easy to see that, under this isomorphism, $\tau_{\phi,\nu}$ is the canonical map ${}^wR_{\mu,+} \to {}^wR_{\mu,+}/I$. The last claim in the lemma follows from Remark 2.13(b). □

Now, let $v \in I_{\nu,-}$ and $w = w_\nu v^{-1}$, so $w \in I^\nu_{\phi,+}$. Recall that ${}^v\mathbf{O}_{\nu,-} = \mathbf{mod}({}^vR_{\nu,-})$ and ${}^v\mathbf{O}_{\phi,-} = \mathbf{mod}({}^vR_{\phi,-})$. We equip the k-algebras ${}^vR_{\nu,-}$ and ${}^vR_{\phi,-}$ with the Koszul gradings ${}^w\bar{R}^\nu_{\phi,+}$ and ${}^w\bar{R}_{\phi,+}$, see Proposition 4.4. Let $\bar{L}_\nu$, $\bar{L}_\phi$ be the natural graded lifts of L_ν, L_ϕ in the graded categories

$${}^v\bar{\mathbf{O}}_{\nu,-} = \mathbf{gmod}({}^w\bar{R}^\nu_{\phi,+}), \quad {}^v\bar{\mathbf{O}}_{\phi,-} = \mathbf{gmod}({}^w\bar{R}_{\phi,+}).$$

Lemma 4.7. *For $v \in I_{\nu,-}$ and $d + m > f$ there is an exact graded functor $\bar{T}_{\phi,\nu}$ such that the square* (4.4) *is commutative and $\bar{T}_{\phi,\nu}(\bar{L}_\phi) = \bar{L}_\nu$*

$$\begin{tikzcd}
{}^v\bar{\mathbf{O}}_{\phi,-} \arrow[r, "\bar{T}_{\phi,\nu}"'] \arrow[d] & {}^v\bar{\mathbf{O}}_{\nu,-} \arrow[d] \\
{}^v\mathbf{O}_{\phi,-} \arrow[r, "T_{\phi,\nu}"] & {}^v\mathbf{O}_{\nu,-}.
\end{tikzcd} \tag{4.4}$$

Proof. First, note that the functor $T_{\phi,\nu} : {}^v\mathbf{O}_{\phi,-} \to {}^v\mathbf{O}_{\nu,-}$ is well-defined, because $d + m > f$ and $v \in I^{\max}_\nu$. By Remark 3.36 we have

$${}^v\mathbf{O}_{\nu,-} = \mathbf{mod}({}^vA_{\nu,-}), \quad {}^v\mathbf{O}_{\phi,-} = \mathbf{mod}({}^vA_{\phi,-}).$$

The functors $\mathbb{V}$ on ${}^v\mathbf{O}^\Delta_{\nu,-}$ and ${}^v\mathbf{O}^\Delta_{\phi,-}$ are right exact, and under the identifications above we have $\mathbb{V}({}^vA_{\nu,-}) = {}^vB_{\nu,-}$ and $\mathbb{V}({}^vA_{\phi,-}) = {}^vB_{\phi,-}$. Thus we have respectively

$$\mathbb{V}(M) = {}^vB_{\nu,-} \otimes_{{}^vA_{\nu,-}} M, \quad \mathbb{V}(M) = {}^vB_{\phi,-} \otimes_{{}^vA_{\phi,-}} M.$$

Next, by Corollary 4.5 we have graded k-algebra isomorphisms

$$ {}^w\bar{R}^\nu_{\phi,+} = {}^v\bar{A}_{\nu,-}, \quad {}^w\bar{R}_{\phi,+} = {}^v\bar{A}_{\phi,-}. $$

Therefore, we have the commutative square

$$\begin{tikzcd} {}^v\bar{\mathbf{O}}^\Delta_{\nu,-} \arrow[r, "\bar{\mathbb{V}}"'] \arrow[d] & {}^v\bar{\mathbf{Z}}_{\nu,-} \arrow[d] \\ {}^v\mathbf{O}^\Delta_{\nu,-} \arrow[r, "\mathbb{V}"] & {}^v\mathbf{Z}_{\nu,-} \end{tikzcd} \tag{4.5}$$

and the commutative square

$$\begin{tikzcd} {}^v\bar{\mathbf{O}}^\Delta_{\phi,-} \arrow[r, "\bar{\mathbb{V}}"'] \arrow[d] & {}^v\bar{\mathbf{Z}}_{\phi,-} \arrow[d] \\ {}^v\mathbf{O}^\Delta_{\phi,-} \arrow[r, "\mathbb{V}"] & {}^v\mathbf{Z}_{\phi,-}, \end{tikzcd} \tag{4.6}$$

where we have set respectively

$$ \bar{\mathbb{V}}(M) = {}^v\bar{B}_{\nu,-} \otimes_{{}^v\bar{A}_{\nu,-}} M, \quad \bar{\mathbb{V}}(M) = {}^v\bar{B}_{\phi,-} \otimes_{{}^v\bar{A}_{\phi,-}} M. $$

Compare (4.2). Let ${}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\phi,-} \subset {}^v\bar{\mathbf{O}}^\Delta_{\phi,-}$ be the full subcategory of the projective objects. We define ${}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\nu,-}$, ${}^v\mathbf{O}^{\mathrm{proj}}_{\nu,-}$ and ${}^v\mathbf{O}^{\mathrm{proj}}_{\phi,-}$ in a similar way. The functor $\bar{\mathbb{V}}$ is fully faithful on ${}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\nu,-}$ and ${}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\phi,-}$, because $\mathbb{V}$ is fully faithful on projectives by Remark 3.36. Therefore, we can identify ${}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\nu,-}$, ${}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\phi,-}$ with some full subcategories ${}^v\bar{\mathbf{Z}}^{\mathrm{proj}}_{\nu,-}$, ${}^v\bar{\mathbf{Z}}^{\mathrm{proj}}_{\phi,-}$ of ${}^v\bar{\mathbf{Z}}_{\nu,-}$, ${}^v\bar{\mathbf{Z}}_{\phi,-}$. By Proposition 3.38($b$) the functor $T_{\phi,\nu}$ gives an exact functor ${}^v\mathbf{O}^{\mathrm{proj}}_{\phi,-} \to {}^v\mathbf{O}^{\mathrm{proj}}_{\nu,-}$. Similarly $\bar{\theta}_{\phi,\nu}$ gives an exact functor ${}^v\bar{\mathbf{Z}}^{\mathrm{proj}}_{\phi,-} \to {}^v\bar{\mathbf{Z}}^{\mathrm{proj}}_{\nu,-}$, see Remark 3.43. Therefore $\bar{\theta}_{\phi,\nu}$ gives an exact functor $\bar{T}_{\phi,\nu} : {}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\phi,-} \to {}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\nu,-}$ via $\bar{\mathbb{V}}$. It coincides with $T_{\phi,\nu}$ when forgetting the grading, by Proposition 3.40(b). Using the same argument as in [3, p. 147] via the homotopy category of projective modules, it gives an exact functor $\bar{T}_{\phi,\nu} : {}^v\bar{\mathbf{O}}_{\phi,-} \to {}^v\bar{\mathbf{O}}_{\nu,-}$ which coincides with $T_{\phi,\nu}$ when forgetting the grading. Thus, the square (4.4) is commutative.

Now, we concentrate on the equality $\bar{T}_{\phi,\nu}(\bar{L}_\phi) = \bar{L}_\nu$. By Proposition 3.38($e$) and (4.4), for $x \in {}^wI_{\nu,-}$ there is an integer j such that

$$ \bar{T}_{\phi,\nu}(\bar{L}(xw_\nu \bullet \mathrm{o}_{\phi,-})) = \bar{L}(x \bullet \mathrm{o}_{\nu,-})\{j\}. $$

We must check that $j = 0$. Recall that ${}^v\bar{P}(x \bullet \mathrm{o}_{\nu,-})$ is the module ${}^vP(x \bullet \mathrm{o}_{\nu,-})$ with its natural grading. Since $\bar{T}_{\nu,\phi}$ is left adjoint to $\bar{T}_{\phi,\nu}$, it is enough to check that

$$ \bar{T}_{\nu,\phi}({}^v\bar{P}(x \bullet \mathrm{o}_{\nu,-})) = {}^v\bar{P}(xw_\nu \bullet \mathrm{o}_{\nu,-}). $$

Now, observe that $\bar{\theta}_{\phi,\nu} \circ \bar{\mathbb{V}}$ is isomorphic to $\bar{\mathbb{V}} \circ \bar{T}_{\phi,\nu}$ on ${}^v\bar{\mathbf{O}}^{\mathrm{proj}}_{\phi,-}$. Therefore, it is enough to check that

$$ \bar{\theta}_{\nu,\phi}({}^v\bar{B}_{\nu,-}(xw_\nu)) = {}^v\bar{B}_{\phi,-}(xw_\nu). $$

This follows from Proposition 3.40(g) by base change. ☐

Remark 4.8. Let $\bar{T}_{\nu,\phi}$ be the left adjoint of the functor $\bar{T}_{\phi,\nu}$. The existence of $\bar{T}_{\nu,\phi}$ follows from general facts as in the proof of Proposition 3.38. By definition of $T_{\nu,\phi}$ and the unicity of the left adjoint, the square (4.7) below is commutative (up to an

isomorphism of functors)

$$\begin{tikzcd} {}^v\bar{\mathbf{O}}_{\phi,-} \arrow[d] & {}^v\bar{\mathbf{O}}_{\nu,-} \arrow[l, "\bar{T}_{\nu,\phi}"'] \arrow[d] \\ {}^v\mathbf{O}_{\phi,-} & {}^v\mathbf{O}_{\nu,-}. \arrow[l, "T_{\nu,\phi}"] \end{tikzcd} \tag{4.7}$$

Lemma 4.9. *For $v \in I_{\nu,-}$ and $d + m > f$ the functor $T_{\phi,\nu} : {}^v\mathbf{O}_{\phi,-} \to {}^v\mathbf{O}_{\nu,-}$ induces a surjective graded k-algebra homomorphism ${}^v\bar{R}^\mu_{\phi,-} \to {}^v\bar{R}^\mu_{\nu,-}$. The kernel of $T_{\phi,\nu}$ contains the two-sided ideal generated by $\{1_x\,;\, x \in {}^vI^\mu_{\phi,-},\ x \notin I_{\nu,+}\}$, and for $x \in {}^vI^\mu_{\phi,-} \cap I_{\nu,+}$ we have $xw_\nu \in {}^vI^\mu_{\nu,-}$ and $T_{\phi,\nu}(1_x) = 1_{xw_\nu}$.*

Proof. First, note that, since $v \in I^{\max}_\nu$ and $d + m > f$, the functor $T_{\phi,\nu} : {}^v\mathbf{O}_{\phi,-} \to {}^v\mathbf{O}_{\nu,-}$ is well-defined and it takes ${}^v\mathbf{O}^\mu_{\phi,-}$ into ${}^v\mathbf{O}^\mu_{\nu,-}$ by Proposition 3.38(c). Composing it with its left adjoint functor $T_{\nu,\phi}$ we get the functor $\Theta = T_{\nu,\phi} \circ T_{\phi,\nu}$. To unburden the notation we abbreviate $L_\phi = {}^vL^\mu_{\phi,-}$ and $L_\nu = {}^vL^\mu_{\nu,-}$. By Proposition 3.38($g$) we have $T_{\phi,\nu}(L_\phi) = L_\nu$. Thus $T_{\phi,\nu}$ induces a graded k-algebra homomorphism ${}^v\bar{R}^\mu_{\phi,-} \to {}^v\bar{R}^\mu_{\nu,-}$. To prove its surjectivity we must prove that the counit $\Theta \to \mathbf{1}$ yields a surjective map

$$\mathrm{Ext}_{{}^v\mathbf{O}^\mu_{\phi,-}}(L_\phi) \to \mathrm{Ext}_{{}^v\mathbf{O}^\mu_{\phi,-}}(\Theta(L_\phi), L_\phi).$$

The parabolic inclusion ${}^v\mathbf{O}^\mu_{\phi,-} \subset {}^v\mathbf{O}_{\phi,-}$ is injective on extensions by Remark 2.13. So we must prove that the counit yields a surjective map

$$\mathrm{Ext}_{{}^v\mathbf{O}_{\phi,-}}(L_\phi) \to \mathrm{Ext}_{{}^v\mathbf{O}_{\phi,-}}(\Theta(L_\phi), L_\phi). \tag{4.8}$$

Now, set $\bar{\Theta} = \bar{T}_{\nu,\phi} \circ \bar{T}_{\phi,\nu}$, where $\bar{T}_{\nu,\phi}$, $\bar{T}_{\phi,\nu}$ are as in Lemma 4.7 and Remark 4.8. For each i we have

$$\mathrm{Ext}^i_{{}^v\mathbf{O}_{\phi,-}}(L_\phi) = \bigoplus_j \mathrm{Ext}^i_{{}^v\bar{\mathbf{O}}_{\phi,-}}(\bar{L}_\phi, \bar{L}_\phi[j]),$$

$$\mathrm{Ext}^i_{{}^v\mathbf{O}_{\phi,-}}(\Theta(L_\phi), L_\phi) = \bigoplus_j \mathrm{Ext}^i_{{}^v\bar{\mathbf{O}}_{\phi,-}}(\bar{\Theta}(\bar{L}_\phi), \bar{L}_\phi[j]).$$

Thus we must prove that for each j the counit $\eta : \bar{\Theta} \to \mathbf{1}$ yields a surjective map

$$\mathrm{Ext}^i_{{}^v\bar{\mathbf{O}}_{\phi,-}}(\bar{L}_\phi, \bar{L}_\phi[j]) \to \mathrm{Ext}^i_{{}^v\bar{\mathbf{O}}_{\phi,-}}(\bar{\Theta}(\bar{L}_\phi), \bar{L}_\phi[j]). \tag{4.9}$$

By Lemma 4.7 we have $\bar{T}_{\phi,\nu}(\bar{L}_\phi) = \bar{L}_\nu$. Further, the gradings on ${}^v\bar{\mathbf{O}}_{\phi,-}$ and ${}^v\bar{\mathbf{O}}_{\nu,-}$ are Koszul by Proposition 4.4. Thus the right hand side of (4.9), which is equal to $\mathrm{Ext}^i_{{}^v\bar{\mathbf{O}}_{\nu,-}}(\bar{L}_\nu, \bar{L}_\nu[j])$, is zero unless $i = j$. Now, let

$$\mathrm{low} = \max\{d\,;\, \bar{\Theta}(\bar{L}_\phi)^d \neq 0\}.$$

Recall that ${}^v\bar{\mathbf{O}}_{\phi,-} = \mathbf{gmod}({}^w\bar{R}_{\phi,+})$, and that the grading on ${}^w\bar{R}_{\phi,+}$ is positive. Thus $\bar{\Theta}(\bar{L}_\phi)^{\mathrm{low}}$ is a quotient module of $\bar{\Theta}(\bar{L}_\phi)$ which is killed by the radical of ${}^w\bar{R}_{\phi,+}$, i.e., we have

$$\bar{\Theta}(\bar{L}_\phi)^{\mathrm{low}} \subset \mathrm{top}(\bar{\Theta}(\bar{L}_\phi)).$$

Next, we claim that for any simple module $\bar{L}$ such that $\bar{T}_{\phi,\nu}(\bar{L}) \neq 0$, the map $\eta(\bar{L})$ factors to an isomorphism $\mathrm{top}(\bar{\Theta}(\bar{L})) \to \bar{L}$. Indeed, $\eta(\bar{L})$ is surjective because it is non zero, and for any simple quotient $\bar{\Theta}(\bar{L}) \to \bar{L}'$ we have

$$0 \neq \mathrm{Hom}_{{}^v\bar{\mathbf{O}}_{\phi,-}}(\bar{\Theta}(\bar{L}), \bar{L}') = \mathrm{Hom}_{{}^v\bar{\mathbf{O}}_\nu}(\bar{T}_{\phi,\nu}(\bar{L}), \bar{T}_{\phi,\nu}(\bar{L}')).$$

By Proposition 3.38(e), (f) this implies that

$$0 \neq \bar{T}_{\phi,\nu}(\bar{L}) = \bar{T}_{\phi,\nu}(\bar{L}').$$

Therefore, we have $\bar{L} = \bar{L}'$. This proves the claim. Applying this claim to any simple direct summand $\bar{L} \subset \bar{L}_\phi$ such that $\bar{T}_{\phi,\nu}(\bar{L}) \neq 0$, we get that $\mathrm{top}(\bar{\Theta}(\bar{L}_\phi)) =$

$\mathrm{Im}\,\eta(\bar{L}_\phi)$. In particular $\mathrm{top}(\bar{\Theta}(\bar{L}_\phi))$ is pure of degree zero. This implies that low $=$ 0. Therefore, the kernel of $\eta(\bar{L}_\phi)$ lives in degrees > 0. Hence the surjectivity of (4.9) for $i = j$ follows from the long exact sequence of Ext's groups and the vanishing

$$\mathrm{Ext}^i_{{}^v\bar{\mathbf{O}}_{\phi,-}}\big(\mathrm{Ker}\,\eta(\bar{L}_\phi), \bar{L}_\phi[i]\big) = 0$$

by Koszulity of the grading of ${}^v\bar{\mathbf{O}}_{\phi,-}$. This proves the first part of the lemma.

Next, note that $I^\mu_{\nu,-} = \{xw_\nu\ ;\ x \in I^\mu_{\phi,-} \cap I_{\nu,+}\}$ by Appendix C, that $T_{\phi,\nu}(1_{xw_\nu}) =$ 0 for $x \notin I_{\nu,-}$ by Proposition 3.38(f), and that $xw_\nu \in I_{\nu,+}$ if and only if $x \in I_{\nu,-}$. This proves the second claim of the lemma. Finally, the last claim of the lemma follows from Proposition 3.38(e). $\square$

Lemma 4.10. *Let $w \in I^\nu_{\mu,+}$ and $v = w_-^{-1}$, so $v \in I^\mu_{\nu,-}$. Assume that $d+m > f$ and $e+m > d$. There is a k-algebra homomorphism $p_{\mu,\nu}$ such that the square* (4.10) *is commutative and $p_{\mu,\nu}(1_x) = 1_y$ for $x \in {}^wI^\nu_{\mu,+}$ and $y = x_-^{-1}$*

$$\begin{tikzcd}
{}^{w_\nu w}R_{\mu,+} \arrow[r, equal, "4.1"'] \arrow[d, "\tau_{\phi,\nu}"] & {}^v\bar{R}^\mu_{\phi,-} \arrow[d, "T_{\phi,\nu}"'] \\
{}^wR^\nu_{\mu,+} \arrow[r, equal, "p_{\mu,\nu}"] & {}^v\bar{R}^\mu_{\nu,-}.
\end{tikzcd} \tag{4.10}$$

Proof. Note that $v = w_\mu w^{-1} w_\nu$, that $w_\nu w \in I^\nu_{\phi,-} \cap I_{\mu,+}$ by Appendix C, and that ${}^{w_\nu w}R^\nu_{\mu,+} = {}^wR^\nu_{\mu,+}$. Now, let $\pi_{\mu,\nu} : {}^{w_\nu w}R_{\mu,+} \to {}^v\bar{R}^\mu_{\nu,-}$ be the composition of $T_{\phi,\nu}$ and the isomorphism ${}^{w_\nu w}R_{\mu,+} = {}^v\bar{R}^\mu_{\phi,-}$ in Proposition 4.1. We must construct a k-algebra isomorphism $p_{\mu,\nu}$ such that $\pi_{\mu,\nu} = p_{\mu,\nu} \circ \tau_{\phi,\nu}$. Let $x \in {}^{w_\nu w}I_{\mu,+}$. Then $w_\mu x^{-1} \in {}^vI^\mu_{\phi,-}$. By Lemmas 4.6, 4.9 we have

$$\begin{aligned}
\pi_{\mu,\nu}(1_x) \neq 0 &\iff T_{\phi,\nu}(1_{w_\mu x^{-1}}) \neq 0 \\
&\iff w_\mu x^{-1} \in I^\mu_{\phi,-} \cap I_{\nu,+}, \\
\tau_{\phi,\nu}(1_x) \neq 0 &\iff x \in I^\nu_{\mu,+}.
\end{aligned}$$

Next, by Appendix C, we have

$$\begin{aligned}
w_\mu x^{-1} \in I^\mu_{\phi,-} \cap I_{\nu,+} &\iff x w_\mu \in I^\nu_{\phi,+} \cap I_{\mu,-} \\
&\iff x \in I^\nu_{\mu,+}.
\end{aligned}$$

Hence, we have $\tau_{\phi,\nu}(1_x) = 0$ if and only if $\pi_{\mu,\nu}(1_x) = 0$. Thus $\mathrm{Ker}(\tau_{\phi,\nu}) \subset \mathrm{Ker}(\pi_{\mu,\nu})$, because the left hand side is generated by the 1_x's killed by $\tau_{\phi,\nu}$ and the right hand side contains the 1_x's killed by $T_{\phi,\nu}$. This proves the existence of a k-algebra homomorphism $p_{\mu,\nu}$ such that the diagram above commutes and such that $p_{\mu,\nu}(1_x) = 1_{w_\mu x^{-1}}$ for $x \in {}^wI^\nu_{\mu,+}$. The map $p_{\mu,\nu}$ is surjective by Lemma 4.9.

Now, we prove that $p_{\mu,\nu}$ is also injective. The functor i yields a graded k-algebra homomorphism ${}^v\bar{R}^\mu_{\nu,-} \to {}^v\bar{R}_{\nu,-}$ by Remark 2.13. Consider the diagram

$$\begin{tikzcd}
{}^{w_\nu w}R_{\mu,+} \arrow[r, equal, "4.1"'] \arrow[d, "T_{\mu,\phi}"] & {}^v\bar{R}^\mu_{\phi,-} \arrow[r, "T_{\phi,\nu}"'] \arrow[d, "i"'] & {}^v\bar{R}^\mu_{\nu,-} \arrow[d, "i"'] \\
{}^zR_{\phi,+} \arrow[r, equal, "4.1"] & {}^v\bar{R}_{\phi,-} \arrow[r, "T_{\phi,\nu}"] & {}^v\bar{R}_{\nu,-},
\end{tikzcd} \tag{4.11}$$

where $z = v^{-1} = w_\nu w w_\mu$. Note that $z \in I^\nu_{\mu,-} \subset I^{\max}_\mu$ and that $v \in I^\mu_{\nu,-} \subset I^{\max}_\nu$. Since $d+m > f$ and $e+m > d$, we define $T_{\phi,\mu}$ and $T_{\phi,\nu}$ as in (3.3). Let $T_{\mu,\phi}$ be the left adjoint of $T_{\phi,\mu}$, see Proposition 3.38(a). The right square in (4.11) commutes, because $T_{\phi,\nu}$ and i commute.

Lemma 4.11. *The left square in (4.11) is commutative.*

Now, applying the factorization above both to $\pi_{\mu,\nu}$ and to $\pi_{\mu,\phi}$ we get a diagram

$$\begin{tikzcd} {}^{w_\nu w}R_{\mu,+} \arrow[r, "\tau_{\phi,\nu}"'] \arrow[d, "T_{\mu,\phi}"] & {}^{w}R^{\nu}_{\mu,+} \arrow[r, "p_{\mu,\nu}"'] \arrow[d, "T_{\mu,\phi}"] & {}^{v}\bar{R}^{\mu}_{\nu,-} \arrow[d, "i"'] \\ {}^{z}R_{\phi,+} \arrow[r, "\tau_{\phi,\nu}"] & {}^{ww_\mu}R^{\nu}_{\phi,+} \arrow[r, "p_{\phi,\nu}"] & {}^{v}\bar{R}_{\nu,-}. \end{tikzcd} \tag{4.12}$$

By Lemma 4.11 the outer rectangle in (4.12) is commutative. The left square in (4.12) is commutative, because $T_{\mu,\phi}$ commutes with τ. Thus, since $\tau_{\phi,\nu}$ is surjective, the right square in (4.12) is also commutative. Therefore, since the middle vertical map (4.12) is injective by Remark 3.42, the injectivity of $p_{\mu,\nu}$ follows from the injectivity of $p_{\phi,\nu}$. Now, note that $p_{\phi,\nu}$ is invertible. Indeed, it is surjective by the discussion above and $\dim({}^{w}R^{\nu}_{\phi,+}) = \dim({}^{v}\bar{R}_{\nu,-})$ by Proposition 4.4.

Now, to prove the last claim in Lemma 4.10 it suffices to observe that the square of maps in (4.10) gives the following diagram for any x in ${}^{w}I^{\nu}_{\mu,+}$

$$\begin{tikzcd} 1_x \arrow[r, mapsto, "4.1"] \arrow[d, mapsto, "\tau_{\phi,\nu}"] & 1_{w_\mu x^{-1}} \arrow[d, mapsto, "T_{\phi,\nu}"'] \\ 1_x \arrow[r, mapsto, "p_{\mu,\nu}"] & 1_{w_\mu x^{-1} w_\nu}, \end{tikzcd}$$

see Proposition 4.1 and Lemmas 4.6, 4.9, and that $x_- = w_\nu x w_\mu$. □

Proof of Lemma 4.11. We must prove that the isomorphisms ${}^{w_\nu w}R_{\mu,+} = {}^{v}\bar{R}^{\mu}_{\phi,-}$ and ${}^{w}R_{\phi,+} = {}^{v}\bar{R}_{\phi,-}$ in Proposition 4.1 yield a commutative square

$$\begin{tikzcd} {}^{w_\nu w}R_{\mu,+} \arrow[r, equal, "4.1"'] \arrow[d, "T_{\mu,\phi}"] & {}^{v}\bar{R}^{\mu}_{\phi,-} \arrow[d, "i"'] \\ {}^{z}R_{\phi,+} \arrow[r, equal, "4.1"] & {}^{v}\bar{R}_{\phi,-}. \end{tikzcd} \tag{4.13}$$

By Proposition 3.38(d), the module $T_{\mu,\phi}({}^{w_\nu w}P_{\mu,+})$ is a direct summand in ${}^{z}P_{\phi,+}$. By Proposition 3.40(g) and Remark 3.43, the sheaf $\theta_{\mu,\phi}({}^{w}B_{\mu,+})$ is a direct summand of ${}^{w}B_{\phi,+}$. Further, by Corollary 3.33, Remarks 3.35, 3.43 and Propositions 3.40(b), (g), we have a commutative diagram

$$\begin{tikzcd} {}^{w_\nu w}R_{\mu,+} \arrow[r, equal, "\mathbb{V}"'] \arrow[d, "T_{\mu,\phi}"] & \mathrm{End}_{{}^{w_\nu w}Z_\mu}({}^{w_\nu w}B_{\mu,+})^{\mathrm{op}} \arrow[d, "\theta_{\mu,\phi}"'] \\ {}^{z}R_{\phi,+} \arrow[r, equal, "\mathbb{V}"] & \mathrm{End}_{{}^{z}Z_\phi}({}^{z}B_{\phi,+})^{\mathrm{op}}. \end{tikzcd} \tag{4.14}$$

Next, by Proposition 3.47 and Corollary 3.48, we have a commutative diagram

$$\begin{tikzcd} \mathrm{End}_{{}^{w_\nu w}Z_\mu}({}^{w_\nu w}B_{\mu,+})^{\mathrm{op}} \arrow[r, equal, "\mathbb{H}"'] \arrow[d, "\theta_{\mu,\phi}"] & {}^{v}\bar{R}^{\mu}_{\phi,-} \arrow[d, "i"'] \\ \mathrm{End}_{{}^{z}Z_\phi}({}^{z}B_{\phi,+})^{\mathrm{op}} \arrow[r, equal, "\mathbb{H}"] & {}^{v}\bar{R}_{\phi,-}. \end{tikzcd} \tag{4.15}$$

Finally, the isomorphism in Proposition 4.1 is the composition of $\mathbb{H}$ and $\mathbb{V}$. □

Proof of Theorem 2.16. First, note that, since the highest weight categories ${}^{w}\mathbf{O}^{\nu}_{\mu,+}$, ${}^{v}\mathbf{O}^{\mu}_{\nu,-}$ do not depend on e, f by Remark 2.11, we can assume that there is a positive integer d such that $d+m > f$ and $e+m > d$. Thus the hypothesis of Lemma 4.10 is satisfied. Now, by Propositions 4.1, 4.4 the k-algebra ${}^{w}R_{\mu,\pm}$ has a Koszul grading. Thus, by Lemma 2.2 and Remark 2.13, the k-algebra ${}^{w}R^{\nu}_{\mu,\pm}$ has also a

Koszul grading. By Lemma 2.1 we have $E({}^w R^\nu_{\mu,\pm}) = {}^w\bar{R}^\nu_{\mu,\pm}$ as graded k-algebras. Therefore the graded k-algebra ${}^w\bar{R}^\nu_{\mu,\pm}$ is Koszul. Now, by Lemma 4.10 we have

$$E({}^w\bar{R}^\nu_{\mu,+}) = {}^w R^\nu_{\mu,+} = {}^v\bar{R}^\mu_{\nu,-}$$

as k-algebras. Thus, by unicity of the Koszul grading we have $E({}^w\bar{R}^\nu_{\mu,+}) = {}^v\bar{R}^\mu_{\nu,-}$ as graded k-algebras. Note that the involutivity of the Koszul duality implies that we have also $E({}^w\bar{R}^\nu_{\mu,-}) = {}^v\bar{R}^\mu_{\nu,+}$ as graded k-algebras, and ${}^w R^\nu_{\mu,-} = {}^v\bar{R}^\mu_{\nu,+}$ as k-algebras. □

Appendix A. Finite codimensional affine Schubert varieties

A.1. **Equivariant perverse sheaves on finite dimensional varieties.** By a scheme we always mean a scheme over k. Let T be a torus. A *T-scheme* is a scheme with an algebraic T-action. Fix a contractible T-space ET with a topologically free T-action. For a T-variety X set $X_T = X \times_T ET$. There are obvious projections

$$p : X \times ET \to X, \quad q : X \times ET \to X_T.$$

Let $\mathbf{D}^b_T(X)$ be the T-equivariant bounded derived category. It is the full subcategory of the bounded derived category $\mathbf{D}^b(X_T)$ of sheaves of k-vector spaces on X_T that contains all sheaves $\mathcal{F}$ with an isomorphism $q^*\mathcal{F} \simeq p^*\mathcal{F}_X$ for some $\mathcal{F}_X \in \mathbf{D}^b(X)$. The cohomology of $\mathcal{F} \in \mathbf{D}^b_T(X)$ is the graded S-module

$$H(\mathcal{F}) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{D}^b_T(X)}\big(k_X, \mathcal{F}[i]\big).$$

Here S is identified with $H_T(point)$. Note that for $\mathcal{E}, \mathcal{F} \in \mathbf{D}^b_T(X)$ we have

$$\mathrm{Ext}_{\mathbf{D}^b_T(X)}(\mathcal{E}, \mathcal{F}) = H(\mathcal{RH}om(\mathcal{E}, \mathcal{F})), \tag{A.1}$$

where $\mathcal{RH}om(\mathcal{E}, \mathcal{F})$ is regarded as an object in $\mathbf{D}^b(X_T)$. If $Y \subset X$ is a T-equivariant embedding, let $IC_T(\overline{Y})$ be the minimal extension of $k_Y[\dim Y]$. It is a perverse sheaf on X supported on the zariski closure $\bar{Y}$ in X. Let $IH_T(\bar{Y})$ be the equivariant intersection cohomology of Y and let $H_T(Y)$ be its equivariant cohomology. We have $IH_T(\bar{Y}) = H(IC_T(\bar{Y}))$ and $H_T(Y) = H(k_Y)$. Forgetting the T-action we define $IC(\bar{Y})$ and $IH(\bar{Y})$ in the same way. Now, let X be a quasi-projective T-variety. We say that X is *good* if the following holds

- X has a Whitney stratification $X = \bigsqcup_x X_x$ by T-stable subvarieties,
- $X_x = \mathbb{A}^{l(x)}$ with a linear T-action,
- there are integers $n_{x,y,i} \geqslant 0$ such that

  $$j_y^* IC(\bar{X}_x) = \bigoplus_p k_{X_y}[l(y)][l(x) - l(y) - 2p]^{\oplus n_{x,y,p}} \tag{A.2}$$

  where j_x is the inclusion $X_x \subset X$. Equivalently, we have

  $$j_y^! IC(\bar{X}_x) = \bigoplus_q k_{X_y}[l(y)][l(y) - l(x) + 2q]^{\bigoplus n_{x,y,q}}. \tag{A.3}$$

We call the third property the *parity vanishing*.

Proposition A.1. *If X is a good T-variety then*

(a) $\dim \mathrm{Ext}^i_{\mathbf{D}^b(X)}\big(IC(\bar{X}_x), IC(X_y)\big) = \sum_{z,p,q} n_{x,z,p} n_{y,z,q}$ *where z, p, q runs over the set of triples such that $2l(z) - l(y) - l(x) + 2p + 2q = i$,*

(b) $\mathrm{Ext}_{\mathbf{D}^b(X)}\big(IC(\bar{X}_x), IC(\bar{X}_y)\big) = k\,\mathrm{Ext}_{\mathbf{D}^b_T(X)}\big(IC_T(\bar{X}_x), IC_T(\bar{X}_y)\big)$,

(c) $\mathrm{Ext}_{\mathbf{D}^b_T(X)}\big(IC_T(\bar{X}_x), IC_T(\bar{X}_y)\big) = \mathrm{Hom}_{H_T(X)}\big(IH_T(\bar{X}_x), IH_T(\bar{X}_y)\big)$,

(d) $\mathrm{Ext}_{\mathbf{D}^b(X)}\big(IC(\bar{X}_x), IC(\bar{X}_y)\big) = \mathrm{Hom}_{H(X)}\big(IH(\bar{X}_x), IH(\bar{X}_y)\big)$,

(e) $IH(\bar{X}_x)$ vanishes in degrees $\not\equiv l(x)$ modulo 2 and $IH(\bar{X}_x) = kIH_T(\bar{X}_x)$.

Proof. Part (a) is [5, thm. 3.4.1]. We sketch briefly the proof for the comfort of the reader. Let j_p be the inclusion of $X_p = \bigsqcup_{p=l(x)} X_x$ into X. For $\mathcal{F} \in \mathbf{D}^b(X)$ there is a spectral sequence

$$E_1^{p,q} = H^{p+q}(j_p^! \mathcal{F}) \Rightarrow H^{p+q}(\mathcal{F}).$$

Therefore, if $\mathcal{F} = \mathcal{RH}om\big(IC(\bar{X}_x), IC(\bar{X}_y)\big)$, we get a spectral sequence

$$E_1^{p,q} = \bigoplus_z H^{p+q}\mathcal{RH}om\big(j_z^* IC(\bar{X}_x), j_z^! IC(\bar{X}_y)\big) \Rightarrow \mathrm{Ext}^{p+q}_{\mathbf{D}^b(X)}\big(IC(\bar{X}_x), IC(\bar{X}_y)\big),$$

where z runs over the set of elements with $l(z) = p$. By (A.2), (A.3) the spectral sequence degenerates at E_1, and we get

$$\dim \mathrm{Ext}^i_{\mathbf{D}^b(X)}\big(IC(\bar{X}_x), IC(X_y)\big) = \sum_{z,p,q} n_{x,z,p} n_{y,z,q},$$

where z, p, q are as above. Now, we prove (b). For $\mathcal{F} \in \mathbf{D}^b_T(X)$ there is a spectral sequence [20, sec. 5.5]

$$E_2^{p,q} = S^p \otimes H^q(\mathcal{F}_X) \Rightarrow H^{p+q}(\mathcal{F}).$$

Therefore, if $\mathcal{F} = \mathcal{RH}om\big(IC_T(\bar{X}_x), IC_T(\bar{X}_y)\big)$, we get a spectral sequence

$$E_2^{p,q} = S^p \otimes \mathrm{Ext}^q_{\mathbf{D}^b(X)}(IC(\bar{X}_x), IC(\bar{X}_y)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathbf{D}^b_T(X)}(IC_T(\bar{X}_x), IC_T(\bar{X}_y)).$$

Now $\mathrm{Ext}_{\mathbf{D}^b(X)}(IC(\bar{X}_x), IC(\bar{X}_y))$ vanishes in degrees $\not\equiv l(x) + l(y)$ modulo 2 by (a). Since S vanishes in odd degrees the spectral sequence degenerates at E_2. Thus

$$S \otimes \mathrm{Ext}_{\mathbf{D}^b(X)}(IC(\bar{X}_x), IC(\bar{X}_y)) = \mathrm{Ext}_{\mathbf{D}^b_T(\bar{X})}(IC_T(\bar{X}_x), IC_T(\bar{X}_y)).$$

Part (c) is proved as part (d) below. Compare [17, thm. 5.9]. Part (d) is proved as in [19, sec. 1]. See also [6, sec. 3.3]. Since our setting is slightly different we sketch briefly the main arguments. Fix a partial order on the set of strata such that $\bar{X}_x = X_{\leqslant x} = \bigsqcup_{y \leqslant x} X_y$. Consider the obvious inclusion

$$X_{<x} \xrightarrow{\ i\ } X_{\leqslant x} \xleftarrow{\ j\ } X_x.$$

Let $\mathcal{F}_1 = j^*_{\leqslant x} IC(\bar{X}_x)$ and $\mathcal{F}_2 = j^*_{\leqslant x} IC_T(\bar{X}_y)$, where $j_{\leqslant x}$ is the inclusion $X_{\leqslant x} \subset X$. For any $y \leqslant x$ and $a = 1, 2$ we have

$$j_y^* \mathcal{F}_a = \bigoplus_p k_{X_y}[d_a - 2p]^{\oplus d_{a,y,p}}, \quad j_y^! \mathcal{F}_a = \bigoplus_q k_{X_y}[2l(y) - d_a + 2q]^{\bigoplus d_{a,y,q}}, \quad \text{(A.4)}$$

for some integers d_a and $d_{a,y,i}$. Consider the diagram of graded k-vector spaces

$$\begin{tikzcd}
\mathrm{Ext}_{\mathbf{D}^b(X_{<x})}(i^*\mathcal{F}_1, i^!\mathcal{F}_2) \arrow[r] \arrow[d] & \mathrm{Hom}_{H^*(X_{<x})}(H(i^*\mathcal{F}_1), H(i^!\mathcal{F}_2)) \arrow[d, "a"] \\
\mathrm{Ext}_{\mathbf{D}^b(X_{\leqslant x})}(\mathcal{F}_1, \mathcal{F}_2) \arrow[r] \arrow[d] & \mathrm{Hom}_{H^*(X_{\leqslant x})}(H(\mathcal{F}_1), H(\mathcal{F}_2)) \arrow[d, "b"] \\
\mathrm{Ext}_{\mathbf{D}^b(X_x)}(j^*\mathcal{F}_1, j^*\mathcal{F}_2) \arrow[r] & \mathrm{Hom}_{H^*(X_x)}(H(j^*\mathcal{F}_1), H(j^*\mathcal{F}_2)).
\end{tikzcd}$$

We'll prove that the middle map is invertible by induction on x. The short exact sequence on the left is exact by (A.4), see e.g., [17, lem. 5.3]. For the same reason the lower map is invertible. The complexes $i^*\mathcal{F}_1$ and $i^!\mathcal{F}_2$ on $X_{<x}$ satisfy again (A.4) for any stratum $X_y \subset X_{<x}$. Thus, by induction, we may assume that the upper map is invertible. Then, to prove that the middle map is invertible it is

enough to check that a is injective and that $\mathrm{Im}(a) = \mathrm{Ker}(b)$. Now, by (A.4) we have exact sequences

$$0 \to H(i_* i^! \mathcal{F}_2) \to H(\mathcal{F}_2) \to H(j_* j^* \mathcal{F}_2) \to 0$$
$$0 \to H(j_! j^* \mathcal{F}_1) \to H(\mathcal{F}_1) \to H(i_* i^* \mathcal{F}_1) \to 0,$$

see e.g., [17, lem. 5.3]. Finally, let $\phi \in \mathrm{Ker}(b)$. Then $\mathrm{Im}(\phi)$ lies in $H(i_* i^! \mathcal{F}_2)$. The cohomology with compact support $H_c(X_x)$ fits into an exact sequence

$$0 \to H_c(X_x) \to H(X_{\leqslant x}) \to H(X_{<x}) \to 0.$$

Let $\omega \in H_c(X_x)$ be the fundamental class. We have $\omega \circ \phi = 0$ because $\omega\, H(i_* i^! \mathcal{F}_2) = 0$. Thus we have also $\phi \circ \omega = 0$, i.e., the map ϕ is zero on $H(j_! j^* \mathcal{F}_1) = \omega H(\mathcal{F}_1)$. So ϕ lies in $\mathrm{Im}(a)$. Finally, we prove (e). By parity vanishing the spectral sequence

$$E_1^{p,q} = H^{p+q}(i_p^! IC(\bar{X}_x)) \Rightarrow IH^{p+q}(\bar{X}_x)$$

degenerates. This yields the first claim. The second one follows from the first one, because the spectral sequence below degenerates

$$E_2^{p,q} = S^p \otimes IH^q(\bar{X}_x) \Rightarrow IH^{p+q}(\bar{X}_x).$$

□

A.2. **Equivariant perverse sheaves on infinite dimensional varieties.** Let X be an *essentially smooth* T-scheme, in the sense of [25, sec. 1.6]. Let $\mathbf{D}_T(X)$ be the T-equivariant derived category on X. See [25, sec. 2] for a discussion on the derived category of constructible complexes on X (for the analytic topology) and on perverse sheaves on X. We use the same terminology as in loc. cit. We formulate our results in the T-equivariant setting. The equivariant version of the constructions in [25] is left to the reader. If X, Y are essentially smooth and $Y \to X$ is a T-equivariant embedding of finite presentation, let $IC_T(\overline{Y})$ be the minimal extension of $k_Y[-\operatorname{codim} Y]$ on Y. It is a perverse sheaf on X supported on the zariski closure $\bar{Y}$ in X. Note that the convention for perverse sheaves we use in this section differs from the usual convention for perverse sheaves on finite-dimensional varieties (as in Section A.1 for instance). Usually $k_Y[\dim Y]$ is perverse for a smooth finite-dimensional variety Y. Here $k_Y[-\operatorname{codim} Y]$ is perverse. For $\mathcal{F} \in \mathbf{D}_T(X)$ we define the graded S-module

$$H(\mathcal{F}) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathbf{D}_T(X)}\big(k_X, \mathcal{F}[i]\big). \tag{A.5}$$

For Y as above we abbreviate

$$IH_T(\bar{Y}) = H(IC_T(\bar{Y})), \quad H_T(Y) = H(k_Y).$$

We call $H_T(Y)$ the *equivariant cohomology of* Y. Now, assume that X is the limit of a projective system of smooth schemes

$$X = \varprojlim_n X_n \tag{A.6}$$

as in [25, sec. 1.3]. Let $p_n : X \to X_n$ be the projection and set $Y_n = p_n(Y)$. Assume that Y_n is locally closed in X_n. Since $k_Y = p_n^* k_{Y_n}$ we have an obvious map

$$\begin{aligned} H_T(Y_n) &= \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{D}_T(X_n)}\big(k_{X_n}, k_{Y_n}[i]\big) \\ &\to \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{D}_T(X)}\big(p_n^* k_{X_n}, p_n^* k_{Y_n}[i]\big) \\ &\to \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{D}_T(X)}\big(k_X, k_Y[i]\big) \\ &\to H_T(Y). \end{aligned}$$

It yields an isomorphism

$$H_T(Y) = \varinjlim_n H_T(Y_n).$$

It is clear from (A.5) that $H_T(\mathcal{F})$ is a $H_T(X)$-module. For a morphism $f : Z \to X$ of essentially smooth T-schemes there is a functor $f^* : \mathbf{D}_T(X) \to \mathbf{D}_T(Z)$, see [25, sec. 3.7]. If f is the inclusion of a T-stable subscheme we write $\mathcal{F}_Z = i^*\mathcal{F}$ and

$$IH_T(\overline{Y})_Z = H(IC_T(\overline{Y})_Z).$$

A.3. **The Kashiwara flag manifold.** Let P_μ be the "parabolic subgroup" corresponding to the Lie algebra $\mathbf{p}_\mu$. Let $X = X_\mu = \mathcal{G}/P_\mu$ be the Kashiwara partial flag manifold associated with $\mathbf{g}$ and $\mathbf{p}_\mu$, see [22]. Here $\mathcal{G}$ is the schematic analogue of $G(k((t)))$ defined in [22], which has a locally free right action of the group-k-scheme P_μ and a locally free left action of the group-k-scheme B^-, the "Borel subgroup" opposit to B. Recall that X is an essentially smooth, not quasi-compact, T-scheme, which is covered by T-stable quasi-compact open subsets isomorphic to $\mathbb{A}^\infty = \mathrm{Spec}\, k[x_k\, ;\, k \in \mathbb{N}]$. Let $e_X = P_\mu/P_\mu$ be the origin of X. For $x \in I_{\mu,+}$ we set

$$X^x = B^- x_X = B^- x P_\mu / P_\mu, \quad x_X = x e_X.$$

Note that X^x is a locally closed T-stable subscheme of X of codimension $l(x)$ which is isomorphic to $\mathbb{A}^\infty$. Consider the T-stable subschemes

$$\overline{X^x} = X^{\geqslant x} = \bigsqcup_{y \geqslant x} X^y, \quad X^{\leqslant x} = \bigsqcup_{y\leqslant x} X^y, \quad X^{<x} = \bigsqcup_{y<x} X^y.$$

We call $\overline{X^x}$ a *finite-codimensional affine Schubert variety.* We call $X^{\leqslant x}$ an *admissible open set.* If Ω is an admissible open set, there are canonical isomorphisms

$$IC_T(\overline{X^x})_\Omega = IC_T(\overline{X^x} \cap \Omega), \quad IH_T(\overline{X^x})_\Omega = IH_T(\overline{X^x} \cap \Omega).$$

We can view Ω as the limit of a projective system of smooth schemes (Ω_n) as in [25, lem. 4.4.3]. So, the projection $p_n : \Omega \to \Omega_n$ is a good quotient by a congruence subgroup B_n^- of B^-. Let n be large enough. Then

$$IC_T(\overline{X^x} \cap \Omega) = p_n^* IC_T(\overline{X^x_{\Omega,n}})$$

with $\overline{X^x_{\Omega,n}} = p_n(\overline{X^x} \cap \Omega)$ by [25, sec. 2.6]. Thus we have a map

$$\begin{aligned} IH_T(\overline{X^x_{\Omega,n}}) &= \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{D}_T(\Omega_n)}\big(k_{\Omega_n}, IC_T(\overline{X^x_{\Omega,n}})[i]\big) \\ &\to \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbf{D}_T(\Omega)}\big(k_\Omega, p_n^* IC_T(\overline{X^x_{\Omega,n}})[i]\big) \\ &\to IH_T(\overline{X^x} \cap \Omega) \end{aligned}$$

which yields an isomorphism

$$IH_T(\overline{X^x} \cap \Omega) = \varinjlim_n IH_T(\overline{X^x_{\Omega,n}}).$$

For $\Omega = X^{\leqslant w}$ and $x \leqslant w$ we abbreviate $X^{[x,w]} = \overline{X^x} \cap X^{\leqslant w}$ and $X_n^{[x,w]} = p_n(X^{[x,w]})$. Since p_n is a good quotient by B_n^- and since X^x is B_n^--stable, we have an algebraic stratification

$$X_n^{\leqslant w} = \bigsqcup_{x \leqslant w} X_n^x,$$

where X_n^x is an affine space whose Zarisky closure is $X_n^{[x,w]}$.

Lemma A.2. *(a) The T-variety $X_n^{\leqslant w}$ is smooth and good.*

(b) It is covered by T-stable open affine subsets with an attractive fixed point. The fixed points subset is naturally identified with ${}^wI_{\mu,+}$.

(c) There is a finite number of one-dimensional orbits. The closure of each of them is smooth. Two fixed points are joined by a one-dimensional orbit if and only if the corresponding points in ${}^wI_{\mu,+}$ are joined by an edge in ${}^w\mathcal{G}_\mu$.

Proof. The T-variety $X_n^{\leqslant w}$ is smooth by [25], because $X^{\leqslant w}$ is smooth and p_n is a B_n^--torsor for n large enough. We claim that it is also quasi-projective. Let X_0 be the stack of G-bundles on $\mathbb{P}^1$. We may assume that P_μ is maximal parabolic. Then, by the Drinfeld-Simpson theorem, a k-point of X is the same as a k-point of X_0 with a trivialization of its pullback to $\mathrm{Spec}(k[[t]])$. Here t is regarded as a local coordinate at $\infty \in \mathbb{P}^1$ and we identify B^- with the Iwahori subgroup in $G(k[[t]])$. We may choose B_n^- to be the kernel of the restriction

$$G(k[[t]]) \to G(k[t]/t^n).$$

Then, a k-point of $X_n = X/B_n^-$ is the same as a k-point of X_0 with a trivialization of its pullback to $\mathrm{Spec}(k[t]/t^n)$. We'll prove that there is an increasing system of open subsets $\mathcal{U}_m \subset X_0$ such that for each m and for $n \gg 0$ the fiber product $X_n \times_{X_0} \mathcal{U}_m$ is representable by a quasi-projective variety. This implies our claim.

Choosing a faithful representation $G \subset SL_r$ we can assume that $G = SL_r$. So a k-point of X_0 is the same as a rank r vector bundle on $\mathbb{P}^1$ of degree 0. For an integer $m > 0$ let $\mathcal{U}_m(k)$ be the set of V in $X_0(k)$ with $H^1(\mathbb{P}^1, V \otimes \mathcal{O}(m)) = 0$ which are generated by global sections. It is the set of k-points of an open substack $\mathcal{U}_m$ of X_0. Note that $\mathcal{U}_m \subset \mathcal{U}_{m+1}$ and $X_0 = \bigcup_m \mathcal{U}_m$. Now, the set $\mathcal{Y}_m(k)$ of pairs (V, b) where $V \in \mathcal{U}_m(k)$ and b is a basis of $H^0(\mathbb{P}^1, V \otimes \mathcal{O}(m))$ is the set of k-points of a quasi-projective variety $\mathcal{Y}_m$ by the Grothendieck theory of Quot-schemes. Further, there is a canonical $GL_{r(m+1)}$-action on $\mathcal{Y}_m$ such that the morphism $\mathcal{Y}_m \to \mathcal{U}_m$, $(V, b) \mapsto V$ is a $GL_{r(m+1)}$-bundle. Now, for $n \gg 0$ the fiber product $X_n \times_{X_0} \mathcal{U}_m$ is representable by a quasi-projective variety, see e.g., [42, thm. 5.0.14].

Next, note that $X_n^{\leqslant w}$ is recovered by the open subsets

$$V_n^x = p_n(V^x), \quad x \leqslant w.$$

Each of them contains a unique fixed point under the T-action and finitely many one-dimensional orbits.

Finally, the parity vanishing holds : since $IC(X^{[x,w]}) = p_n^* IC(X_n^{[x,w]})$ we have

$$IC(X_n^{[x,w]})_{X_n^y} = \bigoplus_i k_{X_n^y}[-l(y)][l(y) - l(x) - 2i]^{\oplus Q^{\mu,-1}_{x,y,i}}$$

by [24, thm. 1.3]. The change in the degrees with respect to Section A.1 is due to the change of convention for perverse sheaves mentioned above. □

Now we set $V = \mathbf{t}^*$ and we consider the moment graph ${}^w\mathcal{G}^\vee$.

Proposition A.3. *We have*

(a) $H_T(X^{\leqslant w}) = {}^w\bar{Z}^\vee_{S,\mu}$ and $H(X^{\leqslant w}) = {}^w\bar{Z}^\vee_\mu$ as graded k-algebras,

(b) $IH_T(X^{[x,w]}) = {}^w\bar{B}^\vee_{S,\mu,-}(x)$ as a graded ${}^w\bar{Z}_{S,\mu}$-module,

(c) $IH(X^{[x,w]}) = {}^w\bar{B}^\vee_{\mu,-}(x)$ *as a graded* ${}^w\bar{Z}^\vee_\mu$*-module.*

Proof. Assuming n to be large enough we may assume that

$$H_T(X^{\leqslant w}) = H_T(X_n^{\leqslant w}), \quad IH_T(X^{[x,w]}) = IH_T(X_n^{[x,w]}), \quad \text{etc.}$$

By Lemma A.2 the S-module $H_T(X_n^{\leqslant w})$ is free. Thus we can apply the localization theorem [20, thm. 6.3], which proves (a). Now, we concentrate on (b). The graded k-module $IH(X_n^{[x,w]})$ vanishes in odd degree by Proposition A.1 and Lemma A.2. Thus, applying [7] to $X_n^{[x,w]}$, we get a graded ${}^w\bar{Z}_{S,\mu}$-module isomorphism

$$IH_T^*(X_n^{[x,w]}) = {}^w\bar{B}^\vee_{S,\mu,-}(x).$$

Part (c) follows from (b), Proposition A.1 and Lemma A.2. □

Corollary A.4. *We have a graded S-module isomorphism*

$${}^w\bar{B}^\vee_{S,\mu,-}(x)_y = \bigoplus_{i\geqslant 0} S\{-l(x)-2i\}^{\oplus Q^{\mu,-1}_{x,y,i}}.$$

Proof. Apply Proposition A.3 and [24, thm. 1.3(i)]. □

Proposition A.5. *For each $x, y \leqslant w$ we have*

(a) $\sum_{i\in\mathbb{Z}} t^i \dim k\,\mathrm{Ext}^i_{\mathbf{D}_T(X^{\leqslant w})}\big(IC_T(X^{[x,w]}), IC_T(X^{[y,w]})\big) = \sum_z Q_\mu(t)_{x,z}Q_\mu(t)_{y,z}$,

(b) $\mathrm{Ext}_{\mathbf{D}_T(X^{\leqslant w})}\big(IC_T(X^{[x,w]}), IC_T(X^{[y,w]})\big) = \mathrm{Hom}_{H_T(X^{\leqslant w})}\big(IH_T(X^{[x,w]}), IH_T(X^{[y,w]})\big)$,

(c) $\mathrm{Ext}_{\mathbf{D}(X^{\leqslant w})}\big(IC(X^{[x,w]}), IC(X^{[y,w]})\big) = \mathrm{Hom}_{H(X^{\leqslant w})}\big(IH(X^{[x,w]}), IH(X^{[y,w]})\big)$,

(d) $\mathrm{Ext}_{\mathbf{D}(X^{\leqslant w})}\big(IC(X^{[x,w]}), IC(X^{[y,w]})\big) = k\,\mathrm{Ext}_{\mathbf{D}_T(X^{\leqslant w})}\big(IC_T(X^{[x,w]}), IC_T(X^{[y,w]})\big)$.

Proof. Apply Proposition A.1 and Lemma A.2.. □

Corollary A.6. *We have a graded k-algebra isomorphism*

$${}^w\bar{A}^\vee_{\mu,-} = \mathrm{End}_{{}^wZ^\vee_\mu}\big({}^w\bar{B}^\vee_{\mu,-}\big)^{\mathrm{op}}.$$

Proof. Apply Propositions A.3, A.5. □

Appendix B. Koszul duality and CRDAHA's

B.1. **Combinatorics.** Fix $e, \ell, m > 0$ and set $\mathfrak{g} = \mathfrak{sl}(m)$ and $G = SL(m)$. Let $\mathfrak{b}, \mathfrak{t}$ be the Borel subalgebra of upper triangular matrices and the maximal torus of diagonal matrices. Let (ϵ_i) be the canonical basis of $\mathbb{C}^m$. For any subset $X \subset \mathbb{C}^m$ and any $d \in \mathbb{C}$ we set $X(d) = \big\{(x_1, \dots, x_m) \in X\,;\, \sum_i x_i = d\big\}$. We identify $\mathfrak{t}^* = \mathbb{C}^m/\mathbb{C}\,1^m$, $\mathfrak{t} = \mathbb{C}^m(0)$ and $W = \mathfrak{S}_m$ in the obvious way. Put

$$\rho = (-1, -2, \dots, -m), \quad \alpha_i = \epsilon_i - \epsilon_{i+1}, \quad i \in [1, m).$$

For $\nu \in \mathbb{N}^\ell(m)$, let $\mathbf{p}_\nu \subset \mathbf{g}$ be the parabolic subalgebra with the block diagonal Levi subalgebra the trace free elements in $\mathfrak{gl}(\nu_1) \oplus \cdots \oplus \mathfrak{gl}(\nu_\ell)$. The group W_ν is generated by the simple affine reflections s_i with $i \neq \nu_1, \nu_1+\nu_2, \dots$.

Let $P(n)$ be the set of *partitions of* n. Write $|\lambda| = n$ and let $l(\lambda)$ be the *length* of λ. Let $P(\ell, n)$ be the set of *ℓ-partitions of* n, i.e., the set of ℓ-tuples $\lambda = (\lambda_p)$ of partitions with $\sum_p |\lambda_p| = n$. Finally, let $P(\nu, n) = \{\lambda \in P(\ell, n)\,;\, l(\lambda_p) \leqslant \nu_p\}$. There is an inclusion

$$P(\nu, n) \subset \mathbb{N}^m, \quad \lambda \mapsto \big(\lambda_1 0^{\nu_1 - l(\lambda_1)}, \lambda_2 0^{\nu_2 - l(\lambda_2)}, \dots, \lambda_p 0^{\nu_p - l(\lambda_p)}\big). \tag{B.1}$$

A *node* in a tuple $\lambda \in \mathbb{N}^m$ is a pair of positive integers (x, y) with $x \leqslant m$ and $y \leqslant \lambda_x$. For $\pi \in \mathbb{Z}^m$ we call (x, y) an *(i, π)-node modulo e* if $y - x + \pi_x = i + 1$ modulo e. Let $n_i(\lambda, \pi, e)$ be the number of (i, π)-nodes modulo e in λ.

B.2. **The CRDAHA.** Fix integers $e, \ell, m > 0$. Let $\Gamma \subset \mathbb{C}^\times$ be the group of the ℓ-th roots of 1. Fix $\nu \in \mathbb{Z}^\ell(m)$. We set

$$h = -1/e, \quad h_p = \nu_{p+1}/e - \nu_p/e, \quad p \in \mathbb{Z}/\ell\mathbb{Z}.$$

For $n > 0$ we write $\Gamma_n = \mathfrak{S}_n \ltimes \Gamma^n$. Let $H^\nu(n)$ be the CRDAHA of Γ_n with parameters h and (h_p) and let $\mathcal{O}^\nu(n)$ be its category O. It is a highest weight category with, see [38, sec. 3.3, 3.6], [41, sec. 3.6],

$$\Delta(\mathcal{O}^\nu(n)) = \{\Delta(\lambda)\,;\, \lambda \in P(\ell, n)\}.$$

Write $\Delta^\nu_e(\lambda) = \Delta(\lambda)$. Let $S^\nu_e(\lambda)$ be the top of $\Delta^\nu_e(\lambda)$. We have the block decomposition

$$\mathcal{O}^\nu(n) = \bigoplus_\mu \mathcal{O}^\nu_\mu(n), \tag{B.2}$$

where μ runs over the set of integral weights of $\mathfrak{sl}(e)$. These blocks are determined by the following combinatorial rule. For $\lambda \in P(\ell, n)$ a (i, ν)-*node* in λ is a triple (x, y, p) with $x, y > 0$ and $y \leqslant (\lambda_p)_x$ such that $y - x + \nu_p = i$ modulo e. Let $n^\nu_i(\lambda)$ be the number of (i, ν)-nodes modulo e in λ. Now, we have

$$\Delta^\nu_e(\lambda) \in \mathcal{O}^\nu_\mu(n) \iff |\lambda| = n, \quad \sum_{p=1}^{\ell} \omega_{\nu_p} - \mu = \sum_{i=1}^{e-1} \big(n^\nu_i(\lambda) - n^\nu_0(\lambda)\big)\, \alpha_i. \tag{B.3}$$

The right hand side should be regarded as an equality of integral weights, i.e., the symbols $\omega_1, \omega_2, \dots, \omega_{e-1}$ and $\alpha_1, \alpha_2, \dots, \alpha_{e-1}$ are the fundamental weights and the simple roots of $\mathfrak{sl}(e)$ and μ is a e-tuple modulo $\mathbb{Z}\,1^e$. See [38, lem. 5.16] for details.

An integral weight of $\mathfrak{sl}(e)$ can be represented by an element of $\mathbb{Z}^e(m)$ if and only if it lies in $\omega_m + e\mathbb{Z}\Pi$. Thus, since $\nu \in \mathbb{Z}^\ell(m)$, the integral weights μ which occur in (B.2) or (B.3) can all be represented by e-tuples in $\mathbb{Z}^e(m)$. We re-write the block decomposition in the following way. For $a \in \mathbb{N}$ we set

$$\mathcal{O}^\nu_{\mu,a} = \mathcal{O}^\nu_\mu(n), \quad n = \Big\langle a\delta + \sum_{p=1}^{\ell} \omega_{\nu_p} - \mu : \hat\rho_e \Big\rangle.$$

Then we have

$$\Delta^\nu_e(\lambda) \in \mathcal{O}^\nu_{\mu,a} \iff n^\nu_0(\lambda) = a, \quad \sum_{p=1}^{\ell} \omega_{\nu_p} - \mu = \sum_{i=1}^{e-1} (n^\nu_i(\lambda) - a)\, \alpha_i. \tag{B.4}$$

We are interested by the following conjecture [9, conj. 6].

Conjecture B.1. *The blocks $\mathcal{O}^\nu_{\mu,a}$ and $\mathcal{O}^\mu_{\nu,a}$ are Koszul and are Koszul dual to each other.*

B.3. **The Schur category.** Fix $e, \ell, m > 0$ and $\nu \in \mathbb{N}^\ell(m)$. Set

$$\rho^\nu = (\nu_1, \nu_1 - 1, \dots, 1, \nu_2, \nu_2 - 1, \dots, \nu_\ell, \dots 1).$$

For a ν-dominant integral weight λ of $\mathfrak{sl}(m)$ we abbreviate

$$\lambda^\nu_e = \lambda_e + \rho^\nu - \rho, \quad V^\nu_e(\lambda) = V^\nu(\lambda^\nu_e), \quad L^\nu_e(\lambda) = L(\lambda^\nu_e).$$

Let $\mathbf{S}^\nu(n)$ be the thick subcategory of $\mathbf{O}^\nu$ consisting of the finite length $\mathbf{g}$-modules of level $-e - m$ whose constituents belong to the set $\{L^\nu_e(\lambda)\,;\, \lambda \in P(\nu, n)\}$. The following is proved in [41].

Proposition B.2. *The category $\mathbf{S}^\nu(n)$ is a highest weight category with*

$$\Delta(\mathbf{S}^\nu(n)) = \{V^\nu_e(\lambda)\,;\, \lambda \in P(\nu, n)\}.$$

We have also the following conjecture [41, conj. 8.8].

Conjecture B.3. *There is a quotient functor $\mathcal{O}^\nu(n) \to \mathbf{S}^\nu(n)$ taking $S^\nu_e(\lambda)$ to $L^\nu_e(\lambda)$ if $\lambda \in P(\nu,n)$ and to 0 else. If $P(\nu,n) = P(n)$ this functor is an equivalence of highest weight categories.*

B.4. **Koszul duality of the Schur category.** Fix $e, \ell, m > 0$ and $\nu \in \mathbb{N}^\ell(m)$. Let $\mathbf{S}^\nu_\mu(n)$ be the thick subcategory of $\mathbf{S}^\nu(n)$ containing the modules $L^\nu_e(\lambda)$ such that $S^\nu_e(\lambda) \in \mathcal{O}^\nu_\mu(n)$. By (**??**) and Lemma B.5(a) below, the affine weights λ^ν_e such that $L^\nu_e(\lambda) \in \mathbf{S}^\nu_\mu(n)$ belong to a single linkage class. More precisely, let $\Lambda^\nu_\mu(n)$ the set of ν-dominant integral weights $\lambda \in \mathbb{N}^m$ of $\mathfrak{sl}(m)$ such that

$$|\lambda| = n, \quad \sum_{p=1}^{\ell} \omega_{\nu_p} - \mu = \sum_{i=1}^{e-1} (n^\nu_i(\lambda) - n^\nu_0(\lambda))\, \alpha_i. \tag{B.5}$$

The set $\Lambda^\nu_\mu(n)$ should be regarded as a set of weights of $\mathfrak{sl}(m)$, i.e., each λ should be identified with its class modulo $\mathbb{Z}\,1^m$. Then, we have

$$\mathbf{S}^\nu_\mu(n) = \mathbf{S}^\nu(n) \cap \mathbf{O}^\nu_\mu(n),$$

where $\mathbf{O}^\nu_\mu(n)$ is the thick subcategory of $\mathbf{O}^\nu$ consisting of the finite length modules whose constituents belong to the set $\{L^\nu_e(\lambda)\,;\,\lambda \in \Lambda^\nu_\mu(n)\}$. Finally, set

$$\Lambda^\nu_{\mu,a} = \{\lambda \in \Lambda^\nu_\mu\,;\, n^\nu_0(\lambda) = a\}, \quad \Lambda^\nu_\mu = \bigcup_{n \geqslant 1} \Lambda^\nu_\mu(n), \tag{B.6}$$

and define $\mathbf{O}^\nu_{\mu,a}$ and $\mathbf{S}^\nu_{\mu,a}$ in the obvious way. The following result can be regarded, in view of Conjecture B.3, as an analogue of Conjecture B.1.

Theorem B.4. *The category $\mathbf{S}^\nu_{\mu,a}$ has a Koszul grading. Further, the Koszul dual of $\mathbf{S}^\nu_{\mu,a}$ coincides with the Ringel dual of $\mathbf{S}^\mu_{\nu,a}$.*

Proof. Follows from Theorem 2.16 and Remark 2.9, because, for any $\lambda \in \Lambda^\nu_{\mu,a}$, the stabilizer of λ^ν_e in $\widehat{W}$ for the $\bullet$-action is W_μ by the following lemma. For $\lambda \in \mathbb{N}^m$ we write

$$m_i = n_i(\rho^\nu, -\rho, e),$$
$$k_i(\lambda) = n_i(\lambda + \rho^\nu, -\rho, e),$$
$$k(\lambda) = \big(k_0(\lambda) - k_1(\lambda), k_1(\lambda) - k_2(\lambda), \dots, k_{e-1}(\lambda) - k_0(\lambda) + m\big).$$

Lemma B.5. *For $\lambda, \mu \in \mathbb{N}^m(n)$ the following hold*

(a) $k_i(\lambda) = m_i + n^\nu_i(\lambda)$,

(b) $\lambda^\nu_e \sim \mu^\nu_e$ if and only if $k_i(\lambda) = k_i(\mu)$ for each $i \in \mathbb{Z}/e\mathbb{Z}$,

(c) the stabilizer of λ^ν_e in $\widehat{W}$ for the $\bullet$-action is isomorphic to $W_{k(\lambda)}$ and, if $k_0(\lambda) = m$ and $\lambda^\nu_e \in \mathcal{C}^-$, then this stabilizer is equal to $W_{k^\nu(\lambda)}$.

□

Remark B.6. Consider the particular case $\ell = 1$. Then, by [27], the category $\mathbf{S}^m(n)$ is equivalent to the module category of the q-Schur algebra. Therefore our theorem implies that the q-Schur algebra is Koszul, compare [9]. We get also an explicit description of its Koszul dual in term of affine Lie algebras.

Appendix C. Combinatorics

Recall that $I_{\mu,+} = I^{\min}_\mu$, $I_{\mu,-} = I^{\max}_\mu$ and that

$$I^\nu_{\mu,-} = \{x \in I_{\mu,-}\,;\, x \bullet o_{\mu,-} \text{ is } \nu\text{-dominant}\},$$
$$I^\nu_{\mu,+} = \{x \in I_{\mu,+}\,;\, x \bullet o_{\mu,+} \text{ is } \nu\text{-dominant}\}.$$

Lemma C.1. *We have*

(a) $x \in I^\mu_{\phi,+} \iff x^{-1} \in I_{\mu,+}$,

(b) $x \in I^\mu_{\phi,-} \iff x^{-1} \in I_{\mu,-}$,

(c) $I^\mu_{\phi,\pm} \cap I_{\nu,\mp} = \{xw_\nu\,;\, x \in I^\mu_{\nu,\pm}\}$,

(d) $x \in I^\mu_{\nu,-} \iff w_\mu x w_\nu \in I^\mu_{\nu,+}$.

Proof. To prove (a) note that, since $\mathrm{o}_{\phi,+}$ is dominant regular, we have

$$\begin{aligned}I^\mu_{\phi,+} &= \{x \in \widehat{W}\,;\, x \bullet \mathrm{o}_{\phi,+} \text{ is } \mu\text{-dominant}\}\\ &= \{x \in \widehat{W}\,;\, \langle \mathrm{o}_{\phi,+} + \hat{\rho} : x^{-1}(\check{\alpha})\rangle \geqslant 0,\, \forall \alpha \in \Pi^+_\mu\}\\ &= \{x \in \widehat{W}\,;\, x^{-1}(\Pi^+_\mu) \subset \widehat{\Pi}^+\}\\ &= \{x \in \widehat{W}\,;\, x^{-1} \in I_{\mu,+}\}.\end{aligned}$$

The proof of (b) is similar and is left to the reader. Now we prove part (c). More precisely, choose positive integers d, f such that $\pm(f-d) > -m$. Then the translation functor $T_{\phi,\nu} : {}^z\mathbf{O}_{\phi,\pm} \to {}^z\mathbf{O}_{\nu,\pm}$ in (3.3) is well-defined for any $z \in I^{\max}_\nu$. Thus, by Proposition 3.38$(e),(f),(g)$, we have

$$\begin{aligned}I^\mu_{\nu,\pm} &= \{x\,;\, xw_\nu \in I^\mu_{\phi,\pm},\, x \in I_{\nu,\pm}\}\\ &= \{xw_\nu\,;\, x \in I^\mu_{\phi,\pm},\, x \in I_{\nu,\mp}\}.\end{aligned}$$

Finally, part (d) is standard, see e.g., [8, sec. 2.7]. □

Remark C.2. Therefore, we have

$$I^\mu_{\nu,+} = \{xw_\nu\,;\, x \in (I^{\min}_\mu)^{-1} \cap I^{\max}_\nu\}, \quad I^\mu_{\nu,-} = \{xw_\nu\,;\, x \in (I^{\max}_\mu)^{-1} \cap I^{\min}_\nu\}.$$

Set $I^{\min}_{\mu,\nu} = (I^{\min}_\mu)^{-1} \cap I^{\min}_\nu$ and $I^{\max}_{\mu,\nu} = (I^{\max}_\mu)^{-1} \cap I^{\max}_\nu$. The following inclusions hold, see e.g., [8, sec. 2.7],

$$I^\mu_{\nu,+} \subset I^{\min}_{\mu,\nu}, \quad I^\mu_{\nu,-} \subset I^{\max}_{\mu,\nu}.$$

References

[1] P.N. Achar, S. Kitchen, *Koszul duality and mixed Hodge modules*, arXiv:1105.2181.

[2] I. Agoston, V. Dlab, E. Lukacs, *Quasi-hereditary extension algebras*, Algebr. Represent. Theory **6** (2003), 97-117.

[3] E. Backelin, *Koszul duality for parabolic and singular category $\mathcal{O}$*, Represent. Theory **3** (1999), 139-152.

[4] A. Beilinson, V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves.*

[5] A. Beilinson, V. Ginzburg, W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), 473-527.

[6] R. Bezrukavnikov, Z. Yun, *On Koszul duality for Kac-Moody groups*, arXiv:1101.1253v1.

[7] T. Braden, R. MacPherson, *From moment graphs to intersection cohomology*, Math. Ann. **321** (2001), 533-551.

[8] R. W. Carter, *Finite groups of Lie type, conjugacy classes and complex characters,* Pure and Applied Mathematics (New York), A Wiley-Interscience Publication, John Wiley and Sons, Inc., New York, 1985.

[9] J. Chuang, H. Miyachi, *Hidden Hecke algebras and Koszul duality*, preprint

[10] E. Cline, B.J. Parshall, L. L. Scott *Finite dimensional algebras and highest weight categories*, J. reine. angew. Math. **391** (1988), 85-99.

[11] C. W. Curtis, I. Reiner, *Methods of representation theory, with applications to finite groups and order, I*, Wiley, 1990.

[12] S. Donkin, *The q-Schur algebra*, London Mathematical Society Lecture Note Series, 253. Cambridge University Press, Cambridge, 1998.

[13] P. Fiebig, *Centers and translation functors for the category O over Kac-Moody algebras*, Math. Z. **243** (2003), 689-717.

[14] P. Fiebig, *The combinatorics of category O over symmetrizable Kac-Moody algebras*, Transf. Groups **11** (2006), 26-49.

[15] P. Fiebig, *Sheaves on moment graphs and a localization of Verma flags*, Adv. Math. **217** (2008), 683-712.

[16] P. Fiebig, *The combinatorics of Coxeter categories*, Trans. A.M.S. **360** (2008), 4211-4233.

[17] P. Fiebig, Williamson, G. *Parity sheaves, moment graphs and the p-smooth locus of schubert varieties*, arXiv:1008.0719v1.

[18] E. Frenkel, D. Gaitsgory, *D-modules on the affine grassmannian and representations of affine Kac-Moody algebras*, Duke. Math. J. **125** (2004), 279-327.

[19] V. Ginsburg, *Perverse sheaves and $\mathbb{C}^*$-actions*, J. Amer. Math. Soc. **4** (1991), 483Ð490.

[20] M. Goresky, R. Kottwitz, R. MacPherson, *Equivariant cohomology, Koszul duality and the localization theorem*, Invent. Math. **131** (1998), 25-83.

[21] J. C. Jantzen, *Moment graphs and representations*, Lecture notes, www-fourier.ujf-grenoble.fr/IMG/pdf/notes-jantzen-2.pdf, 2009.

[22] M. Kashiwara, *The flag manifold of Kac-Moody Lie algebra*, *in* "Algebraic Analysis, Geometry and Number theory", Johns Hopkins Press, Baltimore, 1990.

[23] M. Kashiwara, T. Tanisaki, *Characters of irreducible modules with non-critical highest weights over affine Lie algebras,* Representations and quantizations (Shanghai, 1998), 275-296, China High. Educ. Press, Beijing, 2000.

[24] M. Kashiwara, T. Tanisaki, *Parabolic Kazhdan-Lusztig polynomials and Schubert varieties*, J. Algebra **249** (2002), 306-325.

[25] M. Kashiwara, T. Tanisaki, *Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody Lie algebras, II*, in "Operator algebras, unitary representations, enveloping algebras, and invariant theory", Prog. Math., vol. 92, pp. 159-195, Birkhauser, 1990.

[26] M. Kashiwara, T. Tanisaki, *Kazhdan-Lusztig conjecture for affine Lie algebras with negative level*, Duke Math. J. **77** (1995), 21-62.

[27] D. Kazhdan, G., Lusztig, *Tensor structures arising from affine Lie algebras, I-IV*, J. Amer. Math. Soc. **6-7** (1993-1994), 905-947, 949-1011, 335-381, 383-453.

[28] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd Ed., Oxford University Press, 1995.

[29] V. Mazorchuk, S. Ovsienko, *A pairing in homology and the category of linear complexes of tilting modules for a quasi-hereditary algebra*, J. Math. Kyoto Univ. **45** (2005), 711-741.

[30] V. Mazorchuk, *Koszul duality for stratified algebras I. Balanced quasi-hereditary algebras*

[31] V. Mazorchuk, S. Ovsienko, C. Stroppel, *Quadratic duals, Koszul dual functors, and applications* Trans. Amer. Math. soc. **361** (2009), 1129-1172.

[32] B. Mitchell, *Rings with Several Objects*, Adv. Math. **8** (1972), 1-161.

[33] B.J. Parshall, L.L. Scott, *New graded methods in representation theory*, arXiv:0910.0633v2.

[34] S. Riche, *Koszul duality and modular representations of semisimple Lie algebras*, Duke Math. J. **154** (2010), 31-115.

[35] C. M. Ringel, *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences*, Math. Z. **208** (1991), 209-223.

[36] R. Rouquier, *q-Schur algebras and complex reflections groups*, Mosc. Math. J. **8** (2008), 119-158.

[37] P. Shan, *Crystals of Fock spaces and cyclotomic rational double affine Hecke algebras*, Ann. Scient. Éc. Norm. Sup. **44** (2011), 147-182.

[38] P. Shan, E. Vasserot, *Heisenberg algebras and rational double affine Hecke algebras*, arXiv:1011.6488v2.

[39] W. Soergel, *Character formulas for tilting modules over Kac-Moody algebras*, Represent. Theory **2** (1998), 432-448.

[40] C. Stroppel, *TQFT with corners and tilting functors in the Kac-Moody case*, arXiv:math/0605103v1.

[41] M. Varagnolo, E. Vasserot, *Cyclotomic double affine Hecke algebras and affine parabolic category $\mathcal{O}$*, Adv. Math. **225** (2010), 1523-1588.

[42] J. Wang, *The moduli stack of G-bundles*, arXiv:1104.4828v1.

E-mail address: shan@math.jussieu.fr
E-mail address: michela.varagnolo@u-cergy.fr
E-mail address: vasserot@math.jussieu.fr